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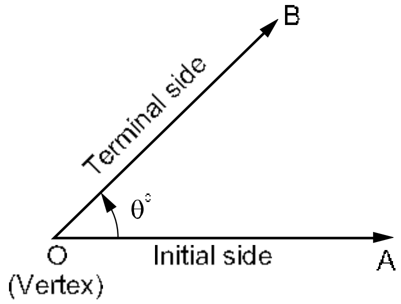
Measurement of Angles

TRIGONOMETRY

The literal meaning of the word *trigonometry* is the science of measuring the sides and the angles of triangles. However, in modern times it is used in a wider context.

We define it *as the branch of mathematics which deals with the measurements of angles of triangles and the problems related to these angles.*

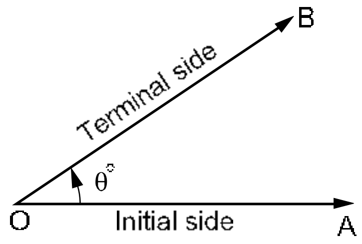
ANGLE When a ray $\overrightarrow{OA}$ starting from its *initial position OA* rotates about its end point *O* and takes the *final position OB*, we say that angle *AOB* (written as $\angle AOB$) has been formed.



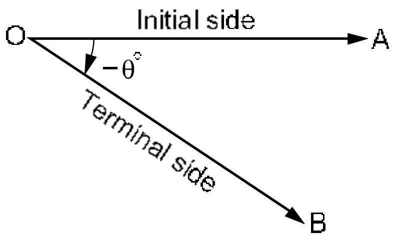
Here, *OA* and *OB* are respectively known as the *initial side* and the *terminal side* of $\angle AOB$, and the point *O* is called its *vertex*.

The amount of rotation from the initial side to the terminal side is called the *measure of the angle*.

POSITIVE AND NEGATIVE ANGLES An angle formed by a rotating ray is said to be positive or negative depending on whether it moves in an anticlockwise or a clockwise direction respectively.



(Positive angle)



(Negative angle)

MEASURING ANGLES Mainly, there are two systems of measuring angles.

(i) Sexagesimal System (Degree Measure)

The angle traced by a moving line about a point from its initial position to the terminating position in making $1/360$ *of the complete revolution of a circle is said to have a measure of* 1 *degree, written as* 1°.

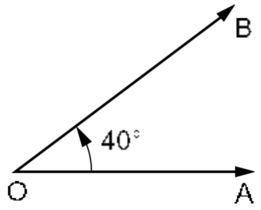
The one-sixtieth part of a degree, i.e., $\frac{1^\circ}{60}$, is called a *minute,* written as $1'$.

The one-sixtieth part of a minute, i.e., $\frac{1'}{60}$, is called a *second,* written as $1''$.

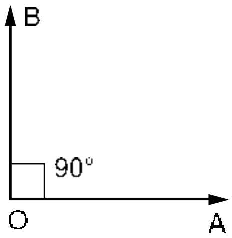
Also, we define 1 right angle = 90°.

An angle measuring 180° is called a *straight angle.*

The measures of the angles given below in (i), (ii), (iii), (iv), (v), (vi) are respectively 40°, 90°, 230°, 420°, –30° and –230°.



(i)



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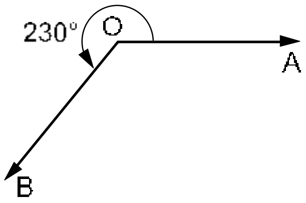
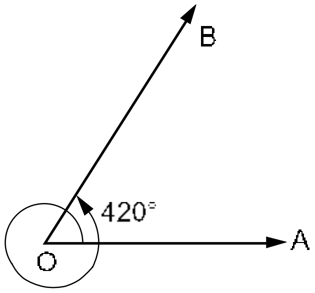
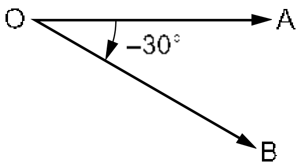


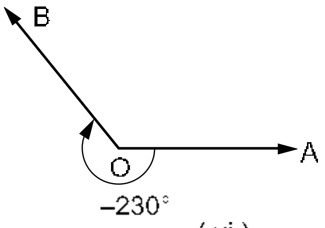
(iii)



(iv)



(v)

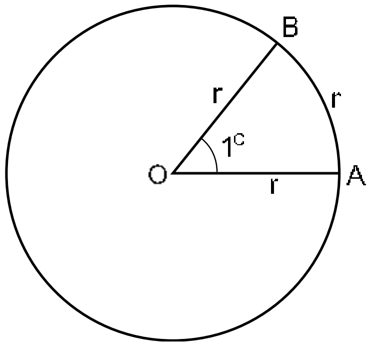


(vi)

(ii) Circular System (Radian Measure)

In this system, an angle is measured in *radians.*

A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

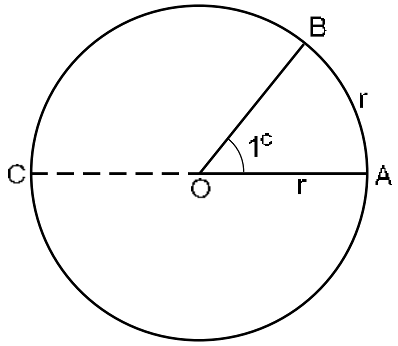


We denote 1 radian by 1^c.

THEOREM 1 *A radian is a constant angle.*

PROOF Let us consider a circle with centre O and radius r.

Let AB be an arc of length r. Join OA and OB.



Then, $\angle AOB = 1^c$.

Produce AO to meet the circle at C.

Then, $\angle AOC$ = a straight angle = 2 rt. ∠s.

Since the angles at the centre of a circle are proportional to the lengths of the arcs subtending them, we have

$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } ABC} = \frac{r}{\frac{1}{2} \cdot (2\pi r)} = \frac{1}{\pi}$$

$$\Rightarrow \angle AOB = \frac{\angle AOC}{\pi} = \frac{2\text{rt}\angle s}{\pi} = \text{constant}$$

$$\Rightarrow 1^c = \text{constant.}$$

Hence, a radian is a constant angle.

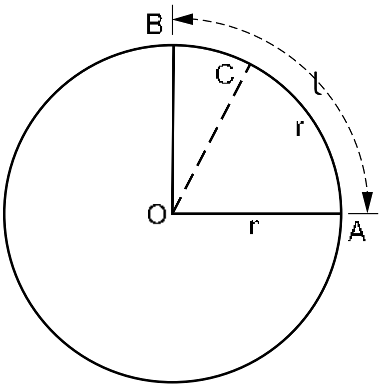
THEOREM 2 *In a circle of radius r, if an arc of length l subtends an angle* θ^c *at the centre then* $\theta = \frac{l}{r}$.

PROOF Let us consider a circle with centre O and radius r.

Let $\angle AOB = \theta^c$ and let arc $AB = l$.

Let C be a point on AB such that arc $AC = r$.

Then, $\angle AOC = 1^c$.



We know that the angles at the centre of a circle are proportional to the lengths of the arcs subtending them.

$$\therefore \frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC} = \frac{l}{r}$$

$$\Rightarrow \frac{\theta}{1} = \frac{l}{r}$$

$$\Rightarrow \theta = \frac{l}{r}.$$

RELATION BETWEEN RADIAN AND DEGREE We know that a complete circle subtends at its centre an angle whose measure is 2π radians as well as $360°$.

$\therefore$ $(2\pi)^c = 360°$.

Hence, $\pi^c = 180°$.

TWO IMPORTANT CONVERSIONS

(i) $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ = \left(\frac{180 \times 7}{22}\right)^\circ = 57°16'\,21''$ (approx.)

(ii) $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c = \left(\frac{22}{7} \times \frac{1}{180}\right)^c = (0.01746)^c$.

Degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

REMARK We express θ^c by the symbol θ only.

SUMMARY

(i) In the sexagesimal system, we measure angles in degrees, minutes and seconds.
1 right angle = 90°, $1^\circ = 60'$ and $1' = 60''$.

(ii) In the circular measure, we measure angles in radians. Thus,

$$\pi^c = 180^\circ.$$

(iii) If an arc of length l makes an angle θ^c at the centre of a circle of radius r, we have

$$\theta = \frac{l}{r}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the degree measure corresponding to each of the following radian measures:*

(i) $\left(\frac{7\pi}{12}\right)^c$ *(ii)* $\left(\frac{3}{4}\right)^c$ *(iii)* $(-2)^c$

SOLUTION (i) $\pi^c = 180^\circ \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow \left(\frac{7\pi}{12}\right)^c = \left(\frac{180}{\pi} \times \frac{7\pi}{12}\right)^\circ = 105^\circ.$$

Hence, $\left(\frac{7\pi}{12}\right)^c = 105^\circ.$

(ii) $\pi^c = 180^\circ \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow \left(\frac{3}{4}\right)^c = \left(\frac{180}{\pi} \times \frac{3}{4}\right)^\circ = \left(180 \times \frac{7}{22} \times \frac{3}{4}\right)^\circ$$

$$= 42^\circ\, 57'\, 16''.$$

$\therefore \left(\frac{3}{4}\right)^c = 42^\circ\, 57'\, 16''.$

(iii) $\pi^c = 180^\circ \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow (-2)^c = \left\{\frac{180}{\pi} \times (-2)\right\}^\circ = \left\{180 \times \frac{7}{22} \times (-2)\right\}^\circ$$

$$= -(114^\circ\, 32'\, 44'').$$

$\therefore (-2)^c = -(114^\circ\, 32'\, 44'').$

EXAMPLE 2 *Find the radian measure corresponding to each of the following degree measures:*

(i) 15° *(ii)* 240° *(iii)* $-37^\circ 30'$

SOLUTION (i) $180^\circ = \pi^c \Rightarrow 1^\circ = \left(\dfrac{\pi}{180}\right)^c$

$$\Rightarrow 15^\circ = \left(\frac{\pi}{180} \times 15\right)^c = \left(\frac{\pi}{12}\right)^c.$$

Hence, $15^\circ = \left(\dfrac{\pi}{12}\right)^c$.

(ii) $180^\circ = \pi^c \Rightarrow 1^\circ = \left(\dfrac{\pi}{180}\right)^c$

$$\Rightarrow 240^\circ = \left(\frac{\pi}{180} \times 240\right)^c = \left(\frac{4\pi}{3}\right)^c.$$

Hence, $240^\circ = \left(\dfrac{4\pi}{3}\right)^c$.

(iii) $180^\circ = \pi^c \Rightarrow 1^\circ = \left(\dfrac{\pi}{180}\right)^c$

$$\Rightarrow -37^\circ 30' = \left\{\frac{\pi}{180} \times \left(\frac{-75}{2}\right)\right\}^c$$

$$\left[-37^\circ 30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ\right]$$

$$= \left(\frac{-5\pi}{24}\right)^c.$$

Hence, $(-37^\circ 30') = \left(\dfrac{-5\pi}{24}\right)^c$.

EXAMPLE 3 *Find in degrees the angle subtended at the centre of a circle of diameter* 50 cm *by an arc of length* 11 cm.

SOLUTION Here, $r = 25$ cm and $l = 11$ cm.

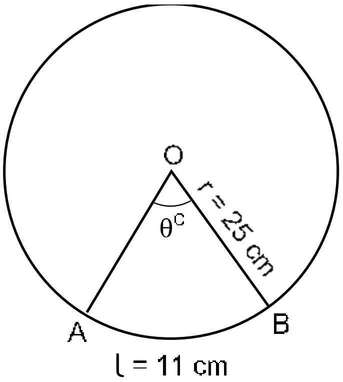
Let the measure of the required angle be θ^c.

Then, $\theta^c = \left(\dfrac{l}{r}\right)^c = \left(\dfrac{11}{25}\right)^c$

$$= \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ \quad [\because \pi^c = 180^\circ]$$

$$= \left(\frac{11}{25} \times \frac{7}{22} \times 180\right)^\circ$$

$$= \left(\frac{126}{5}\right)^\circ = 25^\circ 12'.$$

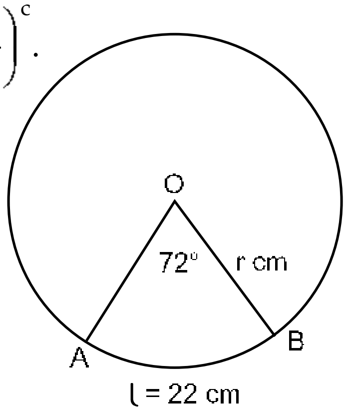


Hence, the required angle is $25^\circ 12'$.

EXAMPLE 4 *Find the radius of a circle in which a central angle of* 72° *intercepts an arc of length* 22 cm. (Use $\pi = 22/7$.)

SOLUTION Let the required radius be r cm. Then,

$$l = 22\,\text{cm and}\ \theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c.$$



$$\text{Now, } \theta = \frac{l}{r} \Rightarrow r = \frac{l}{\theta} = \left(22 \times \frac{5}{2\pi}\right)\text{cm}$$

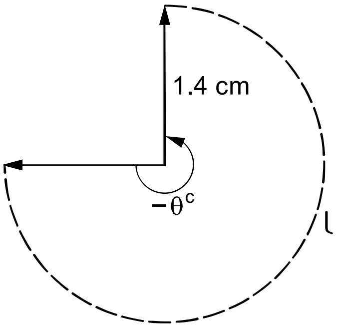
$$= \left(22 \times \frac{5}{2} \times \frac{7}{22}\right)\text{cm}$$

$$= \frac{35}{2}\text{cm} = 17.5\ \text{cm}.$$

Hence, the radius of the circle is 17.5 cm.

EXAMPLE 5 *The minute hand of a watch is* 1.4 cm *long. How far does its tip move in* 45 *minutes?* (Use $\pi = 22/7$.)

SOLUTION In 60 minutes, the minute hand moves through $(2\pi)^c$.



In 45 minutes, the minute hand moves through $\left(\frac{2\pi}{60} \times 45\right)^c = \left(\frac{3\pi}{2}\right)^c$.

$$\therefore \quad r = 1.4\ \text{cm and}\ \theta = \left(\frac{3\pi}{2}\right)^c.$$

Distance moved by the tip of the minute hand in 45 minutes is given by

$$l = r\theta = \left(1.4 \times \frac{3\pi}{2}\right)\text{cm} = \left(1.4 \times \frac{3}{2} \times \frac{22}{7}\right)\text{cm} = 6.6\ \text{cm}.$$

EXAMPLE 6 *If the arcs of the same length in two circles subtend angles of* 60° *and* 75° *at their respective centres, find the ratio of their radii.*

SOLUTION Let r_1 and r_2 be the radii of the two circles. Then,

$$\theta_1 = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c,$$

$$\text{and } \theta_2 = 75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c.$$

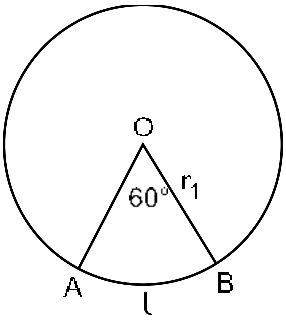
Let the length of each arc be l cm. Then,

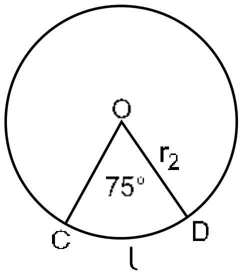
$$l = r_1\theta_1 = r_2\theta_2$$

$$\Rightarrow \left(r_1 \times \frac{\pi}{3}\right) = \left(r_2 \times \frac{5\pi}{12}\right)$$

$$\Rightarrow \frac{r_1}{r_2} = \frac{5}{4}.$$

Hence, $r_1 : r_2 = 5 : 4$.





EXAMPLE 7 *The angles of a triangle are in AP and the ratio of the number of degrees in the least to the number of radians in the greatest is* $60 : \pi$. *Find the angles in degrees and radians.*

SOLUTION Let the angles of the triangle be $(a-d)^\circ$, a° and $(a+d)^\circ$. Then,

$$(a-d)+a+(a+d)=180 \Rightarrow 3a=180 \Rightarrow a=60.$$

Thus, the angles are $(60-d)^\circ$, 60° and $(60+d)^\circ$.

Number of degrees in the least angle $=(60-d)$.

Number of radians in the greatest angle $=\left\{(60+d)\times\frac{\pi}{180}\right\}$.

$$\frac{(60-d)}{(60+d)\times\frac{\pi}{180}}=\frac{60}{\pi} \Leftrightarrow (60-d)\times\pi=60\times(60+d)\times\frac{\pi}{180}$$

$$\Leftrightarrow 3(60-d)=(60+d)$$

$$\Leftrightarrow 4d=120 \Leftrightarrow d=30.$$

$\therefore$ the required angles are $(60-30)^\circ$, 60° and $(60+30)^\circ$, i.e., 30°, 60° and 90°.

These angles in radians are

$$\left(30\times\frac{\pi}{180}\right)^c, \left(60\times\frac{\pi}{180}\right)^c, \left(90\times\frac{\pi}{180}\right)^c, \text{ i.e., } \left(\frac{\pi}{6}\right)^c, \left(\frac{\pi}{3}\right)^c, \left(\frac{\pi}{2}\right)^c.$$

EXAMPLE 8 *In a right-angled triangle, the difference between the two acute angles is* $(\pi/15)^c$. *Find the angles in degrees.*

SOLUTION Clearly, the sum of the two acute angles of a right triangle is 90°.

Difference between the acute angles $=\left(\frac{\pi}{15}\right)^c=\left(\frac{\pi}{15}\times\frac{180}{\pi}\right)^\circ=12^\circ$.

Let the two acute angles be x° and y°. Then,

$$x+y=90 \quad \ldots \text{(i)}$$

$$x-y=12 \quad \ldots \text{(ii)}$$

Solving (i) and (ii), we get $x=51$ and $y=39$.

Hence, the angles of the triangle are 51°, 39° and 90°.

EXAMPLE 9 *A wheel makes* 360 *revolutions in* 1 *minute. Through how many radians does it turn in* 1 *second?*

SOLUTION Number of revolutions made in 60 seconds = 360.

Number of revolutions made in 1 second $=\frac{360}{60}=6$.

Angle moved in 1 revolution $=(2\pi)^c$.

Angle moved in 6 revolutions $=(2\pi\times 6)^c=(12\pi)^c$.

EXAMPLE 10 *Find the angle between the minute hand and the hour hand of a clock when the time is* 7.20.

SOLUTION Angle traced by the hour hand in 12 hours = 360°.

Angle traced by it in 7 h 20 min, i.e., in $\frac{22}{3}$ hours $= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220°$.

Angle traced by the minute hand in 60 min = 360°.

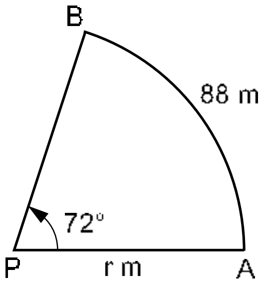
Angle traced by the minute hand in 20 min $\left(\frac{360}{60} \times 20\right)^\circ = 120°$.

Hence, the required angle between the two hands

$$= (220° - 120°) = 100°.$$

EXAMPLE 11 *A horse is tied to a post by a rope. If the horse moves along a circular path, always keeping the rope tight, and describes 88 metres when it traces 72° at the centre, find the length of the rope.*

SOLUTION Let us denote the post by P and let PA be the length of the rope in the tightest position. Suppose that the horse moves along the arc AB so that $\angle APB = 72°$ and arc AB = 88 m.



Let the length of the rope PA be r metres.

Then, $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and l = 88 m.

$\therefore \quad r = \frac{l}{\theta} = \frac{88}{(2\pi/5)}$ m $= \left(88 \times \frac{5}{2} \times \frac{7}{22}\right)$ m $= 70$ m.

Hence, the length of the rope is 70 m.

EXERCISE 14

1. Using a protractor, draw each of the following angles.

   (i) 60° (ii) 130° (iii) 300° (iv) 430°
   (v) –40° (vi) –220° (vii) –310° (viii) –400°

2. Express each of the following angles in radians.

   (i) 36° (ii) 120° (iii) 225° (iv) 330°
   (v) 400° (vi) 7° 30′ (vii) –270° (viii) –(22° 30′)

3. Express each of the following angles in degrees.

   (i) $\left(\frac{5\pi}{12}\right)^c$ (ii) $-\left(\frac{18\pi}{5}\right)^c$ (iii) $\left(\frac{5}{6}\right)^c$ (iv) -4^c

4. The angles of a triangle are in AP and the greatest angle is double the least. Find all the angles in degrees and radians.

5. The difference between the two acute angles of a right triangle is $(\pi/5)^c$. Find these angles in radians and degrees.

6. Find the radius of a circle in which a central angle of 45° intercepts an arc of length 33 cm. (Take $\pi = 22/7$.)

7. Find the length of an arc of a circle of radius 14 cm which subtends an angle of 36° at the centre.

8. If the arcs of the same length in two circles subtend angles $75°$ and $120°$ at the centre, find the ratio of their radii.

9. Find the degree measure of the angle subtended at the centre of a circle of diameter 60 cm by an arc of length 16.5 cm.

10. In a circle of diameter 30 cm, the length of a chord is 15 cm. Find the length of the minor arc of the chord.

11. Find the angle in radians as well as in degrees through which a pendulum swings if its length is 45 cm and its tip describes an arc of length 11 cm.

12. The large hand of a clock is 42 cm long. How many centimetres does its extremity move in 20 minutes?

13. A wheel makes 180 revolutions in 1 minute. Through how many radians does it turn in 1 second?

14. A train is moving on a circular curve of radius 1500 m at the rate of 66 km per hour. Through what angle has it turned in 10 seconds?

15. A wire of length 121 cm is bent so as to lie along the arc of a circle of radius 180 cm. Find in degrees, the angle subtended at the centre by the arc.

16. The angles of a quadrilateral are in AP, and the greatest angle is double the least. Express the least angle in radians.

ANSWERS (EXERCISE 14)

2. (i) $\left(\frac{\pi}{5}\right)^c$ (ii) $\left(\frac{2\pi}{3}\right)^c$ (iii) $\left(\frac{5\pi}{4}\right)^c$ (iv) $\left(\frac{11\pi}{6}\right)^c$ (v) $\left(\frac{20\pi}{9}\right)^c$ (vi) $\left(\frac{\pi}{24}\right)^c$ (vii) $-\left(\frac{3\pi}{2}\right)^c$ (viii) $-\left(\frac{\pi}{8}\right)^c$

3. (i) $75°$ (ii) $-648°$ (iii) $47° 43' 38''$ (iv) $-(229° 5' 27'')$

4. $(40°, 60°, 80°)$, $\left\{\left(\frac{2\pi}{9}\right)^c, \left(\frac{\pi}{3}\right)^c, \left(\frac{4\pi}{9}\right)^c\right\}$ **5.** $(63°, 27°)$, $\left\{\left(\frac{7\pi}{20}\right)^c, \left(\frac{3\pi}{20}\right)^c\right\}$

6. 42 cm **7.** 8.8 cm **8.** 8 : 5 **9.** $31° 30'$ **10.** 15.71 cm

11. $\left(\frac{11}{45}\right)^c$, $14°$ **12.** 88 cm **13.** $(6\pi)^c$ **14.** $7°$ **15.** $38° 30'$ **16.** $\left(\frac{\pi}{3}\right)^c$

HINTS TO SOME SELECTED QUESTIONS

4. Let the required angles be $(a-d)°$, $a°$ and $(a+d)°$.

Then, the sum of the angles of a triangle being $180°$, we have

$$(a-d) + a + (a+d) = 180 \Leftrightarrow 3a = 180 \Leftrightarrow a = 60.$$

Also, $(a+d) = 2(a-d) \Leftrightarrow a = 3d$

$\Leftrightarrow 3d = 60$ $[\because a = 60]$

$\Leftrightarrow d = 20.$

Hence, the angles in degrees are 40°, 60°, 80°.

These angles in radians are $\left(40\times\frac{\pi}{180}\right)^c$, $\left(60\times\frac{\pi}{180}\right)^c$ and $\left(80\times\frac{\pi}{180}\right)^c$, i.e., $\left(\frac{2\pi}{9}\right)^c$, $\left(\frac{\pi}{3}\right)^c$ and $\left(\frac{4\pi}{9}\right)^c$.

5. Let the acute angles of the given triangle be x^c and y^c. Then,

$$x + y = \frac{\pi}{2} \quad \ldots \text{(i)}$$

$$x - y = \frac{\pi}{5} \quad \ldots \text{(ii)}$$

Solving (i) and (ii), we get $x = \frac{7\pi}{20}$ and $y = \frac{3\pi}{20}$.

Thus, the required angles are $\left(\frac{7\pi}{20}\right)^c$ and $\left(\frac{3\pi}{20}\right)^c$.

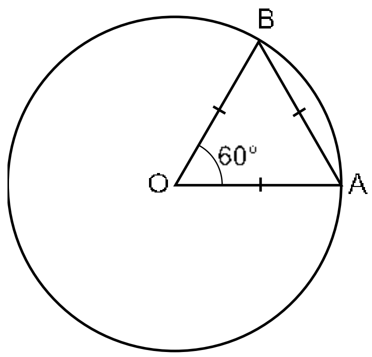
These angles in degrees are $\left(\frac{7\pi}{20}\times\frac{180}{\pi}\right)^\circ$ and $\left(\frac{3\pi}{20}\times\frac{180}{\pi}\right)^\circ$, i.e., 63° and 27°.

10. Let O be the centre of the given circle of radius 15 cm and let AB be the chord of length 15 cm. Then, $OA = OB = AB = 15$ cm.

∴ ΔAOB is an equilateral triangle.

And, therefore, $\theta = \angle AOB = 60^\circ$

$$= \left(60\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c.$$



Let the minor arc $AB = l$ cm. Then,

$$\theta = \left(\frac{\pi}{3}\right)^c \text{ and } r = 15 \text{ cm.}$$

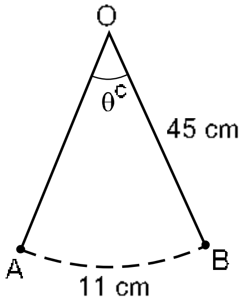
$$\therefore \quad \theta = \frac{l}{r} \Rightarrow l = r\theta = \left(15\times\frac{\pi}{3}\right)\text{cm}$$

$$= \left(5\times\frac{22}{7}\right)\text{cm} = \frac{110}{7}\text{ cm} = 15.71\text{ cm.}$$

11. Here, $r = 45$ cm and $l = 11$ cm.

∴ Let the angle be θ^c. Then,

$$\theta = \frac{l}{r} = \left(\frac{11}{45}\right)^c$$



Now, $\pi^c = 180°$

$$\Rightarrow \left(\frac{11}{45}\right)^c = \left(\frac{180}{\pi}\times\frac{11}{45}\right)^\circ$$

$$= \left(180\times\frac{7}{22}\times\frac{11}{45}\right)^\circ = 14^\circ.$$

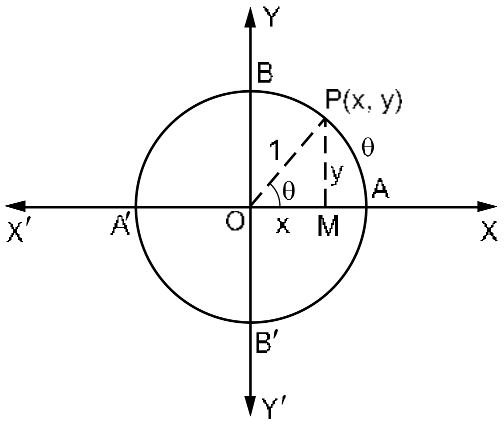
❑

15

Trigonometric, or Circular, Functions

TRIGONOMETRIC (OR CIRCULAR) FUNCTIONS

Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B', as shown in the figure.



Suppose that a moving point starts from A and moves along the circumference of the circle in an anticlockwise direction. Let it cover an arc length θ and take the final position $P(x, y)$. Join OP. Then,

$\angle AOP = \theta$

$$\left[\because \angle AOP = \frac{\text{arc } AP}{\text{radius } OP} = \frac{\theta}{1} = \theta^c, \text{ using } \theta = \frac{l}{r}\right].$$

Now, the six circular functions may be defined as under:

(i) $\cos\theta = x$ (ii) $\sin\theta = y$

(iii) $\sec\theta = \frac{1}{x}, x \neq 0$ (iv) $\operatorname{cosec}\theta = \frac{1}{y}, y \neq 0$

(v) $\tan\theta = \frac{y}{x}, x \neq 0$ (vi) $\cot\theta = \frac{x}{y}, y \neq 0$

REMARK Clearly, we have

(i) $\sec\theta = \frac{1}{\cos\theta}$ (ii) $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

(iii) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (iv) $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

THEOREM 1 *For any acute angle, say* θ, *we have*

(i) $\cos^2\theta + \sin^2\theta = 1$ *(ii)* $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

PROOF Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, meeting OX at A.

Let $P(x, y)$ be a point on the circle with $\angle AOP = \theta$. Join OP.

Draw $PM \perp OA$.

Then, $\cos\theta = x$ and $\sin\theta = y$.

(i) From right $\triangle OMP$, we have

$OM^2 + PM^2 = OP^2$

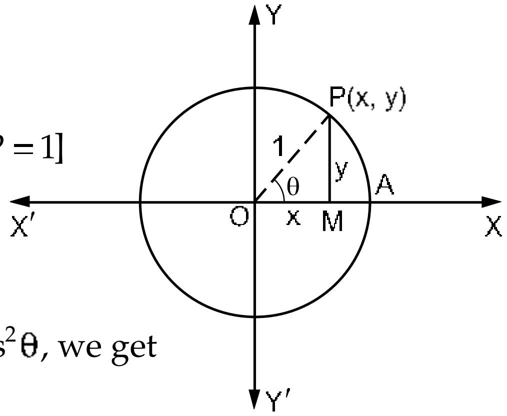
$\Rightarrow \quad x^2 + y^2 = 1$

$[\because OM = x, PM = y \text{ and } OP = 1]$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 1. \ \ldots \text{(i)}$

$[\because x = \cos\theta, y = \sin\theta]$

This proves (i).



(ii) Dividing both sides of (i) by $\cos^2\theta$, we get

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta \qquad \left[\because \frac{\sin\theta}{\cos\theta} = \tan\theta \text{ and } \frac{1}{\cos\theta} = \sec\theta\right].$$

(iii) Dividing both sides of (i) by $\sin^2\theta$, we get

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad \left[\because \frac{\cos\theta}{\sin\theta} = \cot\theta \text{ and } \frac{1}{\sin\theta} = \operatorname{cosec}\theta\right]$$

$$\Rightarrow \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

NEGATIVE ARC LENGTH *If a point moves in a circle then the arc length covered by it is said to be positive or negative depending on whether the point moves in the anticlockwise or clockwise direction respectively.*

<u>THEOREM 2</u> *For any acute angle, say* θ, *we have*

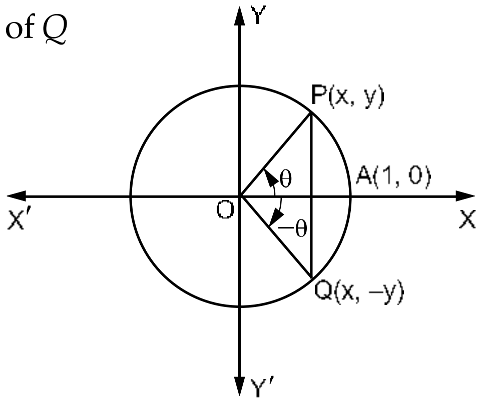
(i) $\cos(-\theta) = \cos\theta$ *(ii)* $\sin(-\theta) = -\sin\theta$ *(iii)* $\tan(-\theta) = -\tan\theta$

(iv) $\cot(-\theta) = -\cot\theta$ *(v)* $\sec(-\theta) = \sec\theta$ *(vi)* $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

PROOF Let $X'OX$ and YOY' be the coordinate axes. With O as the centre, draw a circle of unit radius, meeting OX at $A(1, 0)$.

Let $P(x, y)$ be a point on the circle such that $\angle AOP = \theta$.

Let $\angle AOQ = -\theta$. Then, the coordinates of Q are $Q(x, -y)$.



$\therefore$ (i) $\cos(-\theta) = x = \cos\theta$

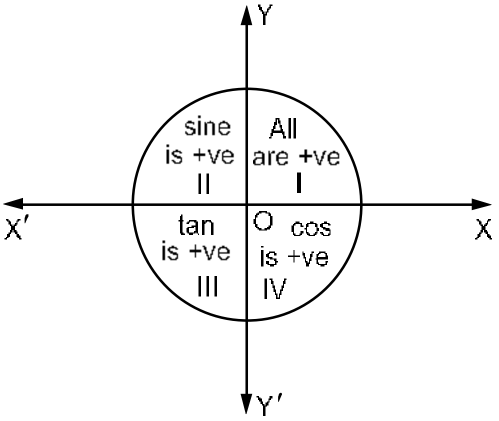
(ii) $\sin(-\theta) = -y = -\sin\theta$

(iii) $$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Working in this order, you can prove the remaining identities by yourself.

Signs of Trigonometric Functions in Various Quadrants

Quadrant	Signs of various T-functions
I	All T-functions are positive.
II	$\sin\theta$ and $\operatorname{cosec}\theta$ are positive. All others are negative.
III	$\tan\theta$ and $\cot\theta$ are positive. All others are negative.
IV	$\cos\theta$ and $\sec\theta$ are positive. All others are negative.



Remember

I	II	III	IV
All are +ve	sin is +ve	tan is +ve	cos is +ve

Values of T-functions of Some Particular Angles

$\theta \to$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

Domain and Range of Trigonometric Functions

The domain and range of each one of the six trigonometric functions is given below.

T-function	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$\{x \in R : x \neq (2n+1)\pi/2, n \in I\}$	R
$\cot x$	$\{x \in R : x \neq n\pi, n \in I\}$	R
$\operatorname{cosec} x$	$\{x \in R : x \neq n\pi, n \in I\}$	$R-]-1, 1[$
$\sec x$	$\{x \in R : x \neq (2n+1)\pi/2, n \in I\}$	$R-]-1, 1[$

PERIODIC FUNCTIONS *A function $f(x)$ is said to be periodic if there exists a constant real number p such that $f(x+p) = f(x)$ for all x.*

The least positive value of p for which $f(x+p) = f(x)$ is called the *period* of $f(x)$.

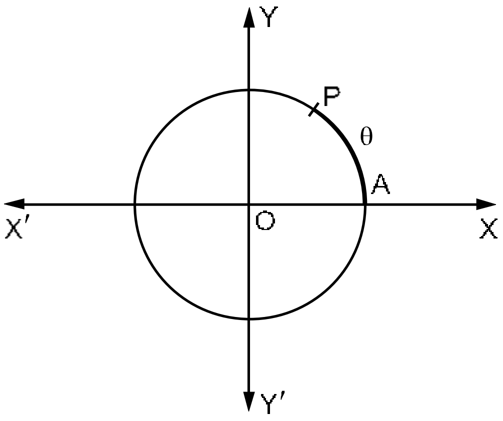
AN IMPORTANT RESULT If $f(x)$ is a periodic function with period p then $f(ax + b)$ with $a > 0$ is a periodic function with period (p/a).

THEOREM 3 $\sin(\theta + 2\pi) = \sin\theta$ *and* $\cos(\theta + 2\pi) = \cos\theta$.

PROOF Let $X'OX$ and YOY' be the coordinate axes. With O as the centre, draw a circle of unit radius, meeting OX at A.

The circumference of this circle of unit radius is 2π.

Let P be a point on this circle such that arc $AP = \theta$.



If we start from the point A, move along the circle and after making a complete revolution, reach P then the length of the arc covered is $(\theta + 2\pi)$.

Since the trigonometric functions are defined in terms of the coordinates of P, we have

$$\cos(\theta + 2\pi) = \cos\theta \text{ and } \sin(\theta + 2\pi) = \sin\theta.$$

REMARKS (i) Since 2π is the least angle for which $\cos(\theta + 2\pi) = \cos\theta$ and $\sin(\theta + 2\pi) = \sin\theta$, it follows that

$\cos\theta$ *and* $\sin\theta$ *are perodic functions, each with period* 2π.

Similarly, $\sec\theta$ *and* $\operatorname{cosec}\theta$ *are periodic, each with period* 2π*; and* $\tan\theta$ *and* $\cot\theta$ *are periodic, each with period* π.

A constant function is periodic, having no period.

(ii) For any real number θ and any integer n, we have:

$$\sin(\theta + 2n\pi) = \sin\theta,\ \cos(\theta + 2n\pi) = \cos\theta$$
$$\tan(\theta + n\pi) = \tan\theta,\ \cot(\theta + n\pi) = \cot\theta$$
$$\sec(\theta + 2n\pi) = \sec\theta,\ \operatorname{cosec}(\theta + 2n\pi) = \operatorname{cosec}\theta$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\sec\theta = \frac{-13}{12}$ *and* θ *lies in the second quadrant, find the values of all the other five trigonometric functions.*

SOLUTION We know that in the second quadrant, $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and all the other trigonometric functions are negative.

Now, $\sec\theta = \frac{-13}{12} \Rightarrow \cos\theta = \frac{1}{\sec\theta} = \frac{-12}{13}$.

$$\therefore \quad \tan^2\theta = (\sec^2\theta - 1) = \left\{\left(\frac{-13}{12}\right)^2 - 1\right\} = \left(\frac{169}{144} - 1\right) = \frac{25}{144}$$

$$\Rightarrow \quad \tan\theta = -\sqrt{\frac{25}{144}} = \frac{-5}{12}$$

$$\Rightarrow \quad \cot\theta = \frac{1}{\tan\theta} = \frac{-12}{5}.$$

Also, $\sin\theta = \tan\theta \cdot \cos\theta = \left(\frac{-5}{12}\right) \times \left(\frac{-12}{13}\right) = \frac{5}{13}.$

And, $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{13}{5}.$

Hence, $\cos\theta = \frac{-12}{13}$; $\tan\theta = \frac{-5}{12}$; $\cot\theta = \frac{-12}{5}$;

$\sin\theta = \frac{5}{13}$ and $\operatorname{cosec}\theta = \frac{13}{5}.$

EXAMPLE 2 *If* $\sin\theta = \frac{-4}{5}$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find the values of all the other five trigonometric functions.*

SOLUTION Clearly, θ lies in the third quadrant in which $\tan\theta$ and $\cot\theta$ are positive and all the other trigonometric functions are negative.

Now, $\sin\theta = \frac{-4}{5} \Rightarrow \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{-5}{4}.$

$$\therefore \quad \cot^2\theta = (\operatorname{cosec}^2\theta - 1) = \left(\frac{25}{16} - 1\right) = \frac{9}{16}$$

$$\Rightarrow \quad \cot\theta = +\sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\cot\theta} = \frac{4}{3}.$$

Also, $\cos\theta = \cot\theta \sin\theta = \left(\frac{3}{4}\right) \times \left(\frac{-4}{5}\right) = \frac{-3}{5}.$

$$\therefore \quad \sec\theta = \frac{1}{\cos\theta} = \frac{-5}{3}.$$

Hence, $\cos\theta = \frac{-3}{5}$; $\tan\theta = \frac{4}{3}$; $\cot\theta = \frac{3}{4}$;

$\sec\theta = \frac{-5}{3}$ and $\operatorname{cosec}\theta = \frac{-5}{4}.$

EXAMPLE 3 *Find the value of*

(i) $\sin\left(\frac{25\pi}{3}\right)$ *(ii)* $\cos\left(\frac{41\pi}{4}\right)$ *(iii)* $\tan\left(\frac{-16\pi}{3}\right)$

(iv) $\cot\left(\frac{29\pi}{4}\right)$ *(v)* $\sec\left(-\frac{19\pi}{3}\right)$ *(vi)* $\operatorname{cosec}\left(-\frac{33\pi}{4}\right)$

SOLUTION (i) $\sin\left(\frac{25\pi}{3}\right) = \sin\left(8\pi + \frac{\pi}{3}\right)$

$= \sin\frac{\pi}{3}$ [$\because$ $\sin(2n\pi + \theta) = \sin\theta$]

$= \frac{\sqrt{3}}{2}$.

(ii) $\cos\left(\frac{41\pi}{4}\right) = \cos\left(10\pi + \frac{\pi}{4}\right)$

$= \cos\frac{\pi}{4}$ [$\because$ $\cos(2n\pi + \theta) = \cos\theta$]

$= \frac{1}{\sqrt{2}}$.

(iii) $\tan\left(-\frac{16\pi}{3}\right) = -\tan\frac{16\pi}{3}$ [$\because$ $\tan(-\theta) = -\tan\theta$]

$= -\tan\left(5\pi + \frac{\pi}{3}\right)$

$= -\tan\frac{\pi}{3}$ [$\because$ $\tan(n\pi + \theta) = \tan\theta$]

$= -\sqrt{3}$.

(iv) $\cot\left(\frac{29\pi}{4}\right) = \cot\left(7\pi + \frac{\pi}{4}\right)$

$= \cot\frac{\pi}{4}$ [$\because$ $\cot(n\pi + \theta) = \cot\theta$]

$= 1$.

(v) $\sec\left(-\frac{19\pi}{3}\right) = \sec\left(\frac{19\pi}{3}\right)$ [$\because$ $\sec(-\theta) = \sec\theta$]

$= \sec\left(6\pi + \frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right)$ [$\because$ $\sec(2n\pi + \theta) = \sec\theta$]

$= 2$.

(vi) $\operatorname{cosec}\left(\frac{-33\pi}{4}\right) = -\operatorname{cosec}\frac{33\pi}{4}$ [$\because$ $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$]

$= -\operatorname{cosec}\left(8\pi + \frac{\pi}{4}\right)$

$= -\operatorname{cosec}\frac{\pi}{4}$ [$\because$ $\operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta$]

$= -\sqrt{2}$.

EXAMPLE 4 *Find the value of*

(i) $\sin(765°)$ *(ii)* $\operatorname{cosec}(-1110°)$ *(iii)* $\cot(-600°)$

SOLUTION (i) $180^\circ = \pi^c$

$$\Rightarrow \quad 765^\circ = \left(\frac{\pi}{180} \times 765\right)^c = \left(\frac{17\pi}{4}\right)^c.$$

$$\therefore \quad \sin(765^\circ) = \sin\left(\frac{17\pi}{4}\right)$$

$$= \sin\left(4\pi + \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} \quad [\because \sin(2n\pi + \theta) = \sin\theta]$$

$$= \frac{1}{\sqrt{2}}.$$

(ii) $180^\circ = \pi^c$

$$\Rightarrow \quad (1110)^\circ = \left(\frac{\pi}{180} \times 1110\right)^c = \left(\frac{37\pi}{6}\right)^c.$$

$$\therefore \quad \operatorname{cosec}(-1110^\circ) = -\operatorname{cosec}(1110^\circ) \quad [\because \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta]$$

$$= -\operatorname{cosec}\left(\frac{37\pi}{6}\right)$$

$$= -\operatorname{cosec}\left(6\pi + \frac{\pi}{6}\right)$$

$$= -\operatorname{cosec}\frac{\pi}{6} \quad [\because \operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta]$$

$$= -2.$$

(iii) $180^\circ = \pi^c$

$$\Rightarrow \quad 600^\circ = \left(\frac{\pi}{180} \times 600\right)^c = \left(\frac{10\pi}{3}\right)^c.$$

$$\therefore \quad \cot(-600^\circ) = -\cot 600^\circ \quad [\because \cot(-\theta) = -\cot\theta]$$

$$= -\cot\left(\frac{10\pi}{3}\right)$$

$$= -\cot\left(3\pi + \frac{\pi}{3}\right)$$

$$= -\cot\frac{\pi}{3} \quad [\because \cot(n\pi + \theta) = \cot\theta]$$

$$= -\frac{1}{\sqrt{3}}.$$

EXAMPLE 5 *Find the value of*

(i) $\cos 15\pi$ *(ii)* $\sin 16\pi$ *(iii)* $\cos(-\pi)$

(iv) $\sin 5\pi$ *(v)* $\tan\left(\frac{5\pi}{4}\right)$ *(vi)* $\sec 6\pi$

SOLUTION (i) $\cos 15\pi = \cos(14\pi + \pi)$

$= \cos\pi$ $\quad [\because \cos(2n\pi + \theta) = \cos\theta]$

$= -1.$

(ii) $\sin 16\pi = \sin(16\pi + 0)$

$= \sin 0^\circ$ $\quad [\because \sin(2n\pi + \theta) = \sin\theta]$

$= 0.$

(iii) $\cos(-\pi) = \cos\pi$ $\quad [\because \cos(-\theta) = \cos\theta]$

$= -1.$

(iv) $\sin 5\pi = \sin(4\pi + \pi)$

$= \sin\pi$ $\quad [\because \sin(2n\pi + \theta) = \sin\theta]$

$= 0.$

(v) $\tan\dfrac{5\pi}{4} = \tan\left(\pi + \dfrac{\pi}{4}\right)$

$= \tan\dfrac{\pi}{4}$ $\quad [\because \tan(n\pi + \theta) = \tan\theta]$

$= 1.$

(vi) $\sec 6\pi = \sec(6\pi + 0)$

$= \sec 0^\circ$ $\quad [\because \sec(2n\pi + \theta) = \sec\theta]$

$= 1.$

EXAMPLE 6 *Prove that* $\sin^2\dfrac{\pi}{6} + \cos^2\dfrac{\pi}{3} - \tan^2\dfrac{\pi}{4} = -\dfrac{1}{2}.$

SOLUTION LHS $= \sin^2\dfrac{\pi}{6} + \cos^2\dfrac{\pi}{3} - \tan^2\dfrac{\pi}{4}$

$= \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2 - 1^2$ $\quad \left[\because \sin\dfrac{\pi}{6} = \dfrac{1}{2}, \cos\dfrac{\pi}{3} = \dfrac{1}{2} \text{ and } \tan\dfrac{\pi}{4} = 1\right]$

$= \left(\dfrac{1}{4} + \dfrac{1}{4} - 1\right) = -\dfrac{1}{2} =$ RHS.

EXERCISE 15A

1. If $\cos\theta = \dfrac{-\sqrt{3}}{2}$ and θ lies in Quadrant III, find the values of all the other five trigonometric functions.

2. If $\sin\theta = \dfrac{-1}{2}$ and θ lies in Quadrant IV, find the values of all the other five trigonometric functions.

3. If $\operatorname{cosec}\theta = \dfrac{5}{3}$ and θ lies in Quadrant II, find the values of all the other five trigonometric functions.

4. If $\sec\theta = \sqrt{2}$ and θ lies in Quadrant IV, find the values of all the other trigonometric functions.

5. If $\sin x = \dfrac{-2\sqrt{6}}{5}$ and x lies in Quadrant III, find the values of $\cos x$ and $\cot x$.

6. If $\cos x = \dfrac{-\sqrt{15}}{4}$ and $\dfrac{\pi}{2} < x < \pi$, find the value of $\sin x$.

7. If $\sec x = -2$ and $\pi < x < \dfrac{3\pi}{2}$, find the values of all the other five trigonometric functions.

8. Find the value of

(i) $\sin\left(\dfrac{31\pi}{3}\right)$ (ii) $\cos\left(\dfrac{17\pi}{2}\right)$ (iii) $\tan\left(\dfrac{-25\pi}{3}\right)$

(iv) $\cot\left(\dfrac{13\pi}{4}\right)$ (v) $\sec\left(\dfrac{-25\pi}{3}\right)$ (vi) $\operatorname{cosec}\left(\dfrac{-41\pi}{4}\right)$

9. Find the value of

(i) $\sin 405°$ (ii) $\sec(-1470°)$ (iii) $\tan(-300°)$

(iv) $\cot(585°)$ (v) $\operatorname{cosec}(-750°)$ (vi) $\cos(-2220°)$

10. Prove that

(i) $\tan^2\dfrac{\pi}{3} + 2\cos^2\dfrac{\pi}{4} + 3\sec^2\dfrac{\pi}{6} + 4\cos^2\dfrac{\pi}{2} = 8$

(ii) $\sin\dfrac{\pi}{6}\cos 0 + \sin\dfrac{\pi}{4}\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{3}\cos\dfrac{\pi}{6} = \dfrac{7}{4}$

(iii) $4\sin\dfrac{\pi}{6}\sin^2\dfrac{\pi}{3} + 3\cos\dfrac{\pi}{3}\tan\dfrac{\pi}{4} + \operatorname{cosec}^2\dfrac{\pi}{2} = 4$

ANSWERS (EXERCISE 15A)

1. $\sin\theta = -\dfrac{1}{2}$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, $\cot\theta = \sqrt{3}$, $\operatorname{cosec}\theta = -2$, $\sec\theta = \dfrac{-2}{\sqrt{3}}$

2. $\cos\theta = \dfrac{\sqrt{3}}{2}$, $\tan\theta = \dfrac{-1}{\sqrt{3}}$, $\cot\theta = -\sqrt{3}$, $\sec\theta = \dfrac{2}{\sqrt{3}}$, $\operatorname{cosec}\theta = -2$

3. $\sin\theta = \dfrac{3}{5}$, $\cos\theta = \dfrac{-4}{5}$, $\tan\theta = \dfrac{-3}{4}$, $\cot\theta = \dfrac{-4}{3}$, $\sec\theta = \dfrac{-5}{4}$

4. $\sin\theta = \dfrac{-1}{\sqrt{2}}$, $\cos\theta = \dfrac{1}{\sqrt{2}}$, $\tan\theta = -1$, $\cot\theta = -1$, $\operatorname{cosec}\theta = -\sqrt{2}$

5. $\cos x = -\frac{1}{5},\ \cot x = \frac{1}{2\sqrt{6}}$ **6.** $\frac{1}{4}$

7. $\sin x = \frac{-\sqrt{3}}{2},\ \cos x = -\frac{1}{2},\ \tan x = \sqrt{3},\ \cot x = \frac{1}{\sqrt{3}},\ \text{cosec}\, x = \frac{-2}{\sqrt{3}}$

8. (i) $\frac{\sqrt{3}}{2}$ (ii) 0 (iii) $-\sqrt{3}$ (iv) 1 (v) 2 (vi) $-\sqrt{2}$

9. (i) $\frac{1}{\sqrt{2}}$ (ii) $\frac{2}{\sqrt{3}}$ (iii) $\sqrt{3}$ (iv) 1 (v) –2 (vi) $\frac{1}{2}$

TRIGONOMETRIC FUNCTIONS OF SUM AND DIFFERENCE OF NUMBERS

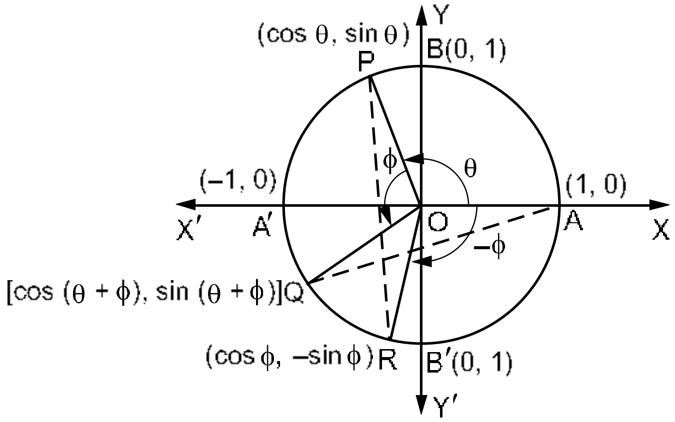
THEOREM 1 *(i)* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

(ii) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

PROOF Let $X'OX$ and YOY' be the coordinate axes. With O as the centre and taking a unit radius, draw a circle, cutting the axes at A, A', B and B' as shown in the figure given below.

Let $\angle AOP = \theta$ and $\angle POQ = \phi$. Then, $\angle AOQ = (\theta + \phi)$.

Let $\angle AOR = -\phi$. Join PR and AQ.



(i) The points on the circle are given by

$P(\cos\theta, \sin\theta), Q[\cos(\theta+\phi), \sin(\theta+\phi)], R[\cos(-\phi), \sin(-\phi)]$,

i.e., $R(\cos\phi, -\sin\phi), A(1, 0), B(0, 1), A'(-1, 0)$ and $B'(0, 1)$.

In ΔPOR and ΔQOA, we have

$PO = QO = 1$ unit, $OR = OA = 1$ unit and

$\angle POR = \angle QOA$ [each equal to $(\phi + \angle QOR)$].

$\therefore\ \Delta POR \cong \Delta QOA$.

Hence, $PR = QA$.

$\therefore\ PR^2 = QA^2$

$\Rightarrow\ (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2$

$= [\cos(\theta+\phi) - 1]^2 + [\sin(\theta+\phi) - 0]^2$

$\Rightarrow$ $(\cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi) + (\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi)$

$= [\cos^2(\theta+\phi) + 1 - 2\cos(\theta+\phi)] + \sin^2(\theta+\phi)$

$\Rightarrow$ $(\cos^2\theta + \sin^2\theta) + (\cos^2\phi + \sin^2\phi) - 2(\cos\theta\cos\phi - \sin\theta\sin\phi)$

$= [\cos^2(\theta+\phi) + \sin^2(\theta+\phi)] + 1 - 2\cos(\theta+\phi)$

$\Rightarrow$ $2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi) = 2 - 2\cos(\theta+\phi)$

$\Rightarrow$ $(\cos\theta\cos\phi - \sin\theta\sin\phi) = \cos(\theta+\phi)$.

Hence, $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. ... (1)

(ii) Replacing ϕ by $-\phi$ in identity (1), we get

$\cos(\theta-\phi) = \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)$.

Hence, $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$[\because \cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi]$.

SUMMARY

(i) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

(ii) $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

THEOREM 2 *(i)* $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ *(ii)* $\left(\frac{\pi}{2} - x\right) = \cos x$

(iii) $\tan\left(\frac{\pi}{2} - x\right) = \cot x$ *(iv)* $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

(v) $\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$ *(vi)* $\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$

PROOF (i) $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= (0 \times \cos x) + (1 \times \sin x)$ $\left[\because \cos\frac{\pi}{2} = 0 \text{ and } \sin\frac{\pi}{2} = 1\right]$

$= \sin x$.

(ii) $\sin\left(\frac{\pi}{2} - x\right) = \sin\theta$, where $\theta = \left(\frac{\pi}{2} - x\right)$

$= \cos\left(\frac{\pi}{2} - \theta\right)$ $\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$

$= \cos\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right\}$

$= \cos x$.

(iii) $\tan\left(\frac{\pi}{2} - x\right) = \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)}$

$= \dfrac{\cos x}{\sin x} = \cot x$ [using (ii) and (i)].

(iv) $\cot\left(\frac{\pi}{2}-x\right)=\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{\sin x}{\cos x}=\tan x$ [using (i) and (ii)].

(v) $\sec\left(\frac{\pi}{2}-x\right)=\frac{1}{\cos\left(\frac{\pi}{2}-x\right)}=\frac{1}{\sin x}=\operatorname{cosec} x\left[\because \cos\left(\frac{\pi}{2}-x\right)=\sin x\right]$

$=\operatorname{cosec} x.$

(vi) $\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\frac{1}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{1}{\cos x}$ $\left[\because \sin\left(\frac{\pi}{2}-x\right)=\cos x\right]$

$=\sec x.$

SUMMARY

(i) $\cos\left(\frac{\pi}{2}-x\right)=\sin x$ **(ii)** $\sin\left(\frac{\pi}{2}-x\right)=\cos x$

(iii) $\tan\left(\frac{\pi}{2}-x\right)=\cot x$ **(iv)** $\cot\left(\frac{\pi}{2}-x\right)=\tan x$

(v) $\sec\left(\frac{\pi}{2}-x\right)=\operatorname{cosec} x$ **(vi)** $\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\sec x$

THEOREM 3 *(i)* $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$ *(ii)* $\sin\left(\frac{\pi}{2}+x\right)=\cos x$

(iii) $\tan\left(\frac{\pi}{2}+x\right)=-\cot x$ *(iv)* $\cot\left(\frac{\pi}{2}+x\right)=-\tan x$

(v) $\sec\left(\frac{\pi}{2}+x\right)=-\operatorname{cosec} x$ *(vi)* $\operatorname{cosec}\left(\frac{\pi}{2}+x\right)=\sec x$

PROOF (i) $\cos\left(\frac{\pi}{2}+x\right)=\cos\frac{\pi}{2}\cos x-\sin\frac{\pi}{2}\sin x$

$=(0\times\cos x)-(1\times\sin x)$ $\left[\because \cos\frac{\pi}{2}=0 \text{ and } \sin\frac{\pi}{2}=1\right]$

$=-\sin x.$

(ii) $\sin\left(\frac{\pi}{2}+x\right)=\sin\left[\frac{\pi}{2}-(-x)\right]$

$=\cos(-x)$ $\left[\because \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta\right]$

$=\cos x$ $[\because \cos(-x)=\cos x]$.

(iii) $\tan\left(\frac{\pi}{2}+x\right) = \frac{\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)}$

$= \frac{\cos x}{-\sin x}$

$\left[\because \sin\left(\frac{\pi}{2}+x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2}+x\right) = -\sin x\right]$

$= -\cot x.$

(iv) $\cot\left(\frac{\pi}{2}+x\right) = \frac{1}{\tan\left(\frac{\pi}{2}+x\right)} = \frac{1}{-\cot x}$ $\quad \left[\because \tan\left(\frac{\pi}{2}+x\right) = -\cot x\right]$

$= -\tan x.$

(v) $\sec\left(\frac{\pi}{2}+x\right) = \frac{1}{\cos\left(\frac{\pi}{2}+x\right)} = \frac{1}{-\sin x}$ $\quad \left[\because \cos\left(\frac{\pi}{2}+x\right) = -\sin x\right]$

$= -\operatorname{cosec} x.$

(vi) $\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{1}{\cos x}$ $\quad \left[\because \sin\left(\frac{\pi}{2}+x\right) = \cos x\right]$

$= \sec x.$

SUMMARY

(i) $\cos\left(\frac{\pi}{2}+x\right) = -\sin x$ **(ii)** $\sin\left(\frac{\pi}{2}+x\right) = \cos x$

(iii) $\tan\left(\frac{\pi}{2}+x\right) = -\cot x$ **(iv)** $\cot\left(\frac{\pi}{2}+x\right) = -\tan x$

(v) $\sec\left(\frac{\pi}{2}+x\right) = -\operatorname{cosec} x$ **(vi)** $\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \sec x$

THEOREM 4 *(i)* $\cos(\pi - x) = -\cos x$ *(ii)* $\sin(\pi - x) = \sin x$

(iii) $\tan(\pi - x) = -\tan x$ *(iv)* $\cot(\pi - x) = -\cot x$

(v) $\sec(\pi - x) = -\sec x$ *(vi)* $\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$

PROOF (i) $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$

$= (-1\times\cos x) + (0\times\sin x)$ $\quad [\because \cos\pi = -1 \text{ and } \sin\pi = 0]$

$= -\cos x.$

(ii) $\sin(\pi - x) = \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} - x\right)\right]$

$= \cos\left(\frac{\pi}{2} - x\right)$ $\quad \left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$= \sin x$ $\quad \left[\because \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$.

(iii) $\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)}$

$= \frac{\sin x}{-\cos x}$ $\quad [\because \sin(\pi - x) = \sin x, \cos(\pi - x) = -\cos x]$

$= -\tan x.$

(iv) $\cot(\pi - x) = \frac{1}{\tan(\pi - x)} = \frac{1}{-\tan x}$ $\quad [\because \tan(\pi - x) = -\tan x]$

$= -\cot x.$

(v) $\sec(\pi - x) = \frac{1}{\cos(\pi - x)} = \frac{1}{-\cos x}$ $\quad [\because \cos(\pi - x) = -\cos x]$

$= -\sec x.$

(vi) $\text{cosec}(\pi - x) = \frac{1}{\sin(\pi - x)} = \frac{1}{\sin x}$ $\quad [\because \sin(\pi - x) = \sin x]$

$= \text{cosec}\, x.$

SUMMARY

(i) $\cos(\pi - x) = -\cos x$	**(ii) $\sin(\pi - x) = \sin x$**
(iii) $\tan(\pi - x) = -\tan x$	**(iv) $\cot(\pi - x) = -\cot x$**
(v) $\sec(\pi - x) = -\sec x$	**(vi) $\text{cosec}(\pi - x) = \text{cosec}\, x$**

THEOREM 5 *(i)* $\cos(\pi + x) = -\cos x$ *(ii)* $\sin(\pi + x) = -\sin x$

(iii) $\tan(\pi + x) = \tan x$ *(iv)* $\cot(\pi + x) = \cot x$

(v) $\sec(\pi + x) = -\sec x$ *(vi)* $\text{cosec}(\pi + x) = -\text{cosec}\, x$

PROOF (i) $\cos(\pi + x) = \cos\pi\cos x - \sin\pi\sin x$

$= (-1)\cdot\cos x - (0 \times \sin x)$ $\quad [\because \cos\pi = -1 \text{ and } \sin\pi = 0]$

$= -\cos x.$

(ii) $\sin(\pi + x) = \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + x\right)\right]$

$= \cos\left(\frac{\pi}{2} + x\right)$ $\quad \left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$= -\sin x.$

(iii) $\tan(\pi + x) = \dfrac{\sin(\pi + x)}{\cos(\pi + x)} = \dfrac{-\sin x}{-\cos x}$

$= \tan x.$ $[\because \sin(\pi + x) = -\sin x, \cos(\pi + x) = -\cos x]$

(iv) $\cot(\pi + x) = \dfrac{1}{\tan(\pi + x)} = \dfrac{1}{\tan x}$ $[\because \tan(\pi + x) = \tan x]$

$= \cot x.$

(v) $\sec(\pi + x) = \dfrac{1}{\cos(\pi + x)} = \dfrac{1}{-\cos x}$ $[\because \cos(\pi + x) = -\cos x]$

$= -\sec x.$

(vi) $\text{cosec}(\pi + x) = \dfrac{1}{\sin(\pi + x)} = \dfrac{1}{-\sin x}$ $[\because \sin(\pi + x) = -\sin x]$

$= -\text{cosec}\, x.$

SUMMARY

(i) $\cos(\pi + x) = -\cos x$	**(ii) $\sin(\pi + x) = -\sin x$**
(iii) $\tan(\pi + x) = \tan x$	**(iv) $\cot(\pi + x) = \cot x$**
(v) $\sec(\pi + x) = -\sec x$	**(vi) $\text{cosec}(\pi + x) = -\text{cosec}\, x$**

THEOREM 6 *(i)* $\cos(2\pi - x) = \cos x$ *(ii)* $\sin(2\pi - x) = -\sin x$
(iii) $\tan(2\pi - x) = -\tan x$ *(iv)* $\cot(2\pi - x) = -\cot x$
(v) $\sec(2\pi - x) = \sec x$ *(vi)* $\text{cosec}(2\pi - x) = -\text{cosec}\, x$

PROOF (i) $\cos(2\pi - x) = \cos 2\pi \cos x + \sin 2\pi \sin x$

$= (1 \times \cos x) + (0 \times \sin x)$ $[\because \cos 2\pi = 1 \text{ and } \sin 2\pi = 0]$

$= \cos x.$

(ii) $\sin(2\pi - x) = \sin[\pi + (\pi + x)]$

$= -\sin(\pi + x)$ $[\because \sin(\pi + \theta) = -\sin\theta]$

$= -[-\sin x] = \sin x.$

The remaining parts are left to the student as an exercise.

SUMMARY

(i) $\cos(2\pi - x) = \cos x$	**(ii) $\sin(2\pi - x) = -\sin x$**
(iii) $\tan(2\pi - x) = -\tan x$	**(iv) $\cot(2\pi - x) = -\cot x$**
(v) $\sec(2\pi - x) = \sec x$	**(vi) $\text{cosec}(2\pi - x) = -\text{cosec}\, x$**

Further, it is easy to verify that

(i) $\cos(2\pi + x) = \cos x$	**(ii) $\sin(2\pi + x) = \sin x$**
(iii) $\tan(2\pi + x) = \tan x$	**(iv) $\cot(2\pi + x) = \cot x$**
(v) $\sec(2\pi + x) = \sec x$	**(vi) $\text{cosec}(2\pi + x) = \text{cosec}\, x$**

Also, we have

(i) $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$	**(ii)** $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
(iii) $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$	**(iv)** $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$

Summarised Table

	$(-x)$	$\left(\frac{\pi}{2} - x\right)$	$\left(\frac{\pi}{2} + x\right)$	$(\pi - x)$	$(\pi + x)$	$\left(\frac{3\pi}{2} - x\right)$	$\left(\frac{3\pi}{2} + x\right)$
sin	$-\sin x$	$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$
cos	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$
tan	$-\tan x$	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$	$\cot x$	$-\cot x$

REMARKS (i) The T-ratios for $(2\pi - x)$ and $(2n\pi - x)$ are the same as those for $(-x)$.
(ii) The T-ratios for $(2\pi + x)$ and $(2n\pi + x)$ are the same as those for x.

THEOREM 7 *(i)* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
(ii) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

PROOF (i) $\sin(\theta + \phi) = \cos\left[\frac{\pi}{2} - (\theta + \phi)\right]$

$$= \cos\left[\left(\frac{\pi}{2} - \theta\right) - \phi\right]$$

$$= \cos\left(\frac{\pi}{2} - \theta\right)\cos\phi + \sin\left(\frac{\pi}{2} - \theta\right)\sin\phi$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Hence, $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

(ii) Replacing ϕ by $-\phi$ in (i), we get

$$\sin[\theta + (-\phi)] = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$$

$$\Rightarrow \sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$[\because \cos(-\phi) = \cos\phi, \sin(-\phi) = -\sin\phi]$.

Hence, $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$.

SUMMARY

(i) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

(ii) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

THEOREM 8 *If none of the angles* θ, ϕ *and* $(\theta + \phi)$ *is an odd multiple of* $\pi/2$ *then*

(i) $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

(ii) $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

PROOF (i) $\tan(\theta + \phi) = \dfrac{\sin(\theta + \phi)}{\cos(\theta + \phi)}$

$$= \frac{(\sin\theta\cos\phi + \cos\theta\sin\phi)}{(\cos\theta\cos\phi - \sin\theta\sin\phi)}$$

$$= \frac{\left(\dfrac{\sin\theta\cos\phi}{\cos\theta\cos\phi} + \dfrac{\cos\theta\sin\phi}{\cos\theta\cos\phi}\right)}{\left(\dfrac{\cos\theta\cos\phi}{\cos\theta\cos\phi} - \dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}\right)}$$

[dividing num. and denom. by $\cos\theta\cos\phi$]

$$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

Hence, $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$.

(ii) Replacing ϕ by $-\phi$ in (i), we get

$$\tan[\theta + (-\phi)] = \frac{\tan\theta + \tan(-\phi)}{1 - \tan\theta\tan(-\phi)}$$

$$\Rightarrow \quad \tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}.$$

Hence, $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$.

THEOREM 9 *If none of the angles* θ, ϕ *and* $(\theta + \phi)$ *is a multiple of* π *then*

(i) $\cot(\theta + \phi) = \dfrac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta}$

(ii) $\cot(\theta - \phi) = \dfrac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}$

PROOF (i) $\cot(\theta + \phi) = \dfrac{\cos(\theta + \phi)}{\sin(\theta + \phi)}$

$$= \frac{(\cos\theta\cos\phi - \sin\theta\sin\phi)}{(\sin\theta\cos\phi + \cos\theta\sin\phi)}$$

$$= \frac{\left(\frac{\cos\theta \cos\phi}{\sin\theta \sin\phi} - \frac{\sin\theta \sin\phi}{\sin\theta \sin\phi}\right)}{\left(\frac{\sin\theta \cos\phi}{\sin\theta \sin\phi} + \frac{\cos\theta \sin\phi}{\sin\theta \sin\phi}\right)}$$

[dividing num. and denom. by $\sin\theta \sin\phi$]

$$= \frac{\cot\theta \cot\phi - 1}{\cot\phi + \cot\theta}.$$

Hence, $\cot(\theta + \phi) = \frac{\cot\theta \cot\phi - 1}{\cot\phi + \cot\theta}$.

(ii) Replacing ϕ by $-\phi$ in (i), we get

$$\cot[\theta + (-\phi)] = \frac{\cot\theta \cot(-\phi) - 1}{\cot(-\phi) + \cot\theta}$$

$$\Rightarrow \cot(\theta - \phi) = \frac{-\cot\theta \cot\phi - 1}{-\cot\phi + \cot\theta} \quad [\because \cot(-\phi) = -\cot\phi]$$

$$\Rightarrow \cot(\theta - \phi) = \frac{\cot\theta \cot\phi + 1}{\cot\phi - \cot\theta}.$$

Hence, $\cot(\theta - \phi) = \frac{\cot\theta \cot\phi + 1}{\cot\phi - \cot\theta}$.

SUMMARY

(i) $\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi}$

(ii) $\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta \tan\phi}$

(iii) $\cot(\theta + \phi) = \frac{\cot\theta \cot\phi - 1}{\cot\phi + \cot\theta}$

(iv) $\cot(\theta - \phi) = \frac{\cot\theta \cot\phi + 1}{\cot\phi - \cot\theta}$

THEOREM 10 *For all $x, y \in R$, we have*

(i) $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

(ii) $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

PROOF (i) $\sin(x + y)\sin(x - y)$

$= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$

$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$

$= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$

$= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y$

$= \sin^2 x - \sin^2 y.$

(ii) $\cos(x+y)\ \cos(x-y)$

$= (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y)$

$= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

$= \cos^2 x(1-\sin^2 y) - (1-\cos^2 x)\sin^2 y$

$= \cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \cos^2 x \sin^2 y$

$= \cos^2 x - \sin^2 y.$

SOLVED EXAMPLES

EXAMPLE 1 *Find the value of*

(i) $\cos 480°$ *(ii)* $\sin 1230°$ *(iii)* $\cot(-135°)$

(iv) $\operatorname{cosec}(-1410°)$ *(v)* $\cos(-870°)$ *(vi)* $\tan 330°$

SOLUTION (i) $\cos 480° = \cos(360° + 120°)$

$= \cos 120°$ $[\because \cos(360° + \theta) = \cos\theta]$

$= \cos(180° - 60°) = -\cos 60°$ $[\because \cos(180° - \theta) = -\cos\theta]$

$= -\frac{1}{2}.$

(ii) $\sin 1230° = \sin(3 \times 360° + 150°)$

$= \sin 150°$ $[\because \sin(2n\pi + \theta) = \sin\theta]$

$= \sin(180° - 30°) = \sin 30°$ $[\because \sin(180° - \theta) = \sin\theta]$

$= \frac{1}{2}.$

(iii) $\cot(-135°) = -\cot 135°$ $[\because \cot(-\theta) = -\cot\theta]$

$= -\cot(90° + 45°) = \tan 45°$

$[\because \cot(90° + \theta) = -\tan\theta]$

$= 1.$

(iv) $\operatorname{cosec}(-1410°) = -\operatorname{cosec} 1410° = -\operatorname{cosec}(4 \times 360° - 30°)$

$= \operatorname{cosec} 30°$ $[\because \operatorname{cosec}(2n\pi - \theta) = -\operatorname{cosec}\theta]$

$= 2.$

(v) $\cos(-870°) = \cos 870°$ $[\because \cos(-\theta) = \cos\theta]$

$= \cos(2 \times 360° + 150°)$

$= \cos 150°$ $[\because \cos(2n\pi + \theta) = \cos\theta]$

$= \cos(180° - 30°) = -\cos 30°$

$[\because \cos(180° - \theta) = -\cos\theta]$

$= \frac{-\sqrt{3}}{2}.$

(vi) $\tan 330° = \tan(360° - 30°) = -\tan 30°$

$[\because \tan(360° - \theta) = -\tan\theta]$

$= -\frac{1}{\sqrt{3}}.$

EXAMPLE 2 *Find the values of all trigonometric functions of (i)* 120° *(ii)* 150°.

SOLUTION (i) $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$;

$\cos 120° = \cos(180° - 60°) = -\cos 60° = \frac{-1}{2}$;

$\tan 120° = \frac{\sin 120°}{\cos 120°} = \left\{\frac{\sqrt{3}}{2} \times (-2)\right\} = -\sqrt{3}$;

$\text{cosec}\, 120° = \frac{1}{\sin 120°} = \frac{2}{\sqrt{3}}$;

$\sec 120° = \frac{1}{\cos 120°} = -2$;

$\cot 120° = \frac{1}{\tan 120°} = \frac{-1}{\sqrt{3}}$.

(ii) $\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$;

$\cos 150° = \cos(180° - 30°) = -\cos 30° = \frac{-\sqrt{3}}{2}$;

$\tan 150° = \frac{\sin 150°}{\cos 150°} = \left(\frac{1}{2} \times \frac{2}{-\sqrt{3}}\right) = \frac{-1}{\sqrt{3}}$;

$\cot 150° = \frac{1}{\tan 150°} = -\sqrt{3}$;

$\sec 150° = \frac{1}{\cos 150°} = \frac{-2}{\sqrt{3}}$;

$\text{cosec}\, 150° = \frac{1}{\sin 150°} = 2$.

EXAMPLE 3 *Show that* $\sin 105° + \cos 105° = \frac{1}{\sqrt{2}}$.

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \sin 105° + \cos 105° \\ &= \sin(60° + 45°) + \cos(60° + 45°) \\ &= (\sin 60° \cos 45° + \cos 60° \sin 45°) \\ &\qquad + (\cos 60° \cos 45° - \sin 60° \sin 45°) \\ &= \left\{\left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)\right\} + \left\{\left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)\right\} \\ &= \left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}} = \text{RHS}.\end{aligned}$$

EXAMPLE 4 *Calculate the value of*

(i) sin 15° *(ii)* cos 15° *(iii)* tan 15°
(iv) sin 75° *(v)* cos 75° *(vi)* tan 75°

SOLUTION (i) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

(ii) $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}.$$

(iii) $\tan 15^\circ = \tan(45^\circ - 30^\circ)$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{\left(1 - \frac{1}{\sqrt{3}}\right)}{1 + \left(1 \times \frac{1}{\sqrt{3}}\right)} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}.$$

(iv) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}.$$

(v) $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right)$$

$$= \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

(vi) $\tan 75^\circ = \tan(45^\circ + 30^\circ)$

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{\left(1 + \frac{1}{\sqrt{3}}\right)}{\left(1 - \frac{1}{\sqrt{3}}\right)}$$

$$= \frac{(\sqrt{3}+1)}{(\sqrt{3}-1)}.$$

EXAMPLE 5 *Evaluate* $\tan \frac{13\pi}{12}$.

SOLUTION We have

$$\tan \frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right)$$

$$= \tan \frac{\pi}{12} \qquad [\because \tan(\pi + \theta) = \tan\theta]$$

$$= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$= \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \frac{(\sqrt{3}-1)^2}{(3-1)}$$

$$= \frac{(3+1-2\sqrt{3})}{2} = \frac{(4-2\sqrt{3})}{2} = (2-\sqrt{3}).$$

EXAMPLE 6 *Prove that*

(i) $\sin 70^\circ \cos 10^\circ - \cos 70^\circ \sin 10^\circ = \frac{\sqrt{3}}{2}$

(ii) $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ = \frac{1}{2}$

(iii) $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ = \frac{1}{2}$

(iv) $\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ = \frac{1}{\sqrt{2}}$

SOLUTION (i) We know that $\sin\theta \cos\phi - \cos\theta \sin\phi = \sin(\theta - \phi)$.

Putting $\theta = 70^\circ$ and $\phi = 10^\circ$ in the this identity, we get

$$\sin 70^\circ \cos 10^\circ - \cos 70^\circ \sin 10^\circ = \sin(70^\circ - 10^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

(ii) We know that $\cos\theta \cos\phi - \sin\theta \sin\phi = \cos(\theta + \phi)$.

Putting $\theta = 50^\circ$ and $\phi = 10^\circ$ in the this identity, we get

$$\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ = \cos(50^\circ + 10^\circ) = \cos 60^\circ = \frac{1}{2}.$$

(iii) We know that $\cos\theta \cos\phi + \sin\theta \sin\phi = \cos(\theta - \phi)$.

Putting $\theta = 80^\circ$ and $\phi = 20^\circ$ in this identity, we get

$$\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ = \cos(80^\circ - 20^\circ) = \cos 60^\circ = \frac{1}{2}.$$

(iv) We know that $\sin\theta \cos\phi + \cos\theta \sin\phi = \sin(\theta + \phi)$.

Putting $\theta = 36^\circ$ and $\phi = 9^\circ$ in this identity, we get

$$\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ = \sin(36^\circ + 9^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}.$$

EXAMPLE 7 *Prove that*

$$\sin(40^\circ + \theta)\cos(10^\circ + \theta) - \cos(40^\circ + \theta)\sin(10^\circ + \theta) = \frac{1}{2}.$$

SOLUTION We know that $\sin x \cos y - \cos x \sin y = \sin(x - y)$.

Putting $x = (40^\circ + \theta)$ and $y = (10^\circ + \theta)$ in this identity, we get

$$\sin(40^\circ + \theta)\cos(10^\circ + \theta) - \cos(40^\circ + \theta)\sin(10^\circ + \theta)$$
$$= \sin[(40^\circ + \theta) - (10^\circ + \theta)]$$
$$= \sin 30^\circ = \frac{1}{2}.$$

EXAMPLE 8 *Prove that*

$$\cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta) = 0.$$

SOLUTION We have

$$\cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta)$$

$$= \cos\theta + \sin\{180^\circ + (90^\circ + \theta)\} - \sin\{180^\circ + (90^\circ - \theta)\} + \cos(180^\circ + \theta)$$

$$= \cos\theta - \sin(90^\circ + \theta) + \sin(90^\circ - \theta) - \cos\theta$$

[$\because$ $\sin(180^\circ + x) = -\sin x$, $\cos(180^\circ + x) = -\cos x$]

$$= \cos\theta - \cos\theta + \cos\theta - \cos\theta = 0$$

[$\because$ $\sin(90^\circ + \theta) = \cos\theta$, $\sin(90^\circ - \theta) = \cos\theta$].

EXAMPLE 9 *Prove that*

$$\frac{\cos(90^\circ + \theta)\sec(270^\circ + \theta)\sin(180^\circ + \theta)}{\operatorname{cosec}(-\theta)\cos(270^\circ - \theta)\tan(180^\circ + \theta)} = \cos\theta.$$

SOLUTION We have

$$\text{LHS} = \frac{\cos(90^\circ + \theta)\sec(270^\circ + \theta)\sin(180^\circ + \theta)}{\operatorname{cosec}(-\theta)\cos(270^\circ - \theta)\tan(180^\circ + \theta)}$$

$$= \frac{\cos(90^\circ + \theta)\sec\{180^\circ + (90^\circ + \theta)\}\sin(180^\circ + \theta)}{-\operatorname{cosec}\theta\cos\{180^\circ + (90^\circ - \theta)\}\tan(180^\circ + \theta)}$$

$$= \frac{\cos(90^\circ + \theta)\{-\sec(90^\circ + \theta)\}\{-\sin\theta\}}{-\operatorname{cosec}\theta\{-\cos(90^\circ - \theta)\}\tan\theta}$$

$$\left[\begin{array}{l}\because \sin(180^\circ + x) = -\sin x,\ \cos(180^\circ + x) = -\cos x \\ \quad \sec(180^\circ + x) = -\sec x,\ \tan(180^\circ + x) = \tan x\end{array}\right]$$

$$= \frac{(-\sin\theta)(\operatorname{cosec}\theta)(-\sin\theta)}{(-\operatorname{cosec}\theta)(-\sin\theta)\tan\theta}$$

[$\because$ $\cos(90^\circ + \theta) = -\sin\theta$, $\sec(90^\circ + \theta) = -\operatorname{cosec}\theta$, $\cos(90^\circ - \theta) = \sin\theta$]

$$= \cos\theta = \text{RHS}.$$

EXAMPLE 10 *Prove that*

$$\cos\left(\frac{3\pi}{2} + \theta\right)\cos(2\pi + \theta)\left[\cot\left(\frac{3\pi}{2} - \theta\right) + \cot(2\pi + \theta)\right] = 1.$$

SOLUTION We have

$$\cos\left(\frac{3\pi}{2} + \theta\right)\cos(2\pi + \theta)\left[\cot\left(\frac{3\pi}{2} - \theta\right) + \cot(2\pi + \theta)\right]$$

$$= \cos\left[\pi + \left(\frac{\pi}{2} + \theta\right)\right]\cos(2\pi + \theta)\left[\cot\left\{\pi + \left(\frac{\pi}{2} - \theta\right)\right\} + \cot(2\pi + \theta)\right]$$

$$= -\cos\left(\frac{\pi}{2} + \theta\right)\cos\theta\left[\cot\left(\frac{\pi}{2} - \theta\right) + \cot\theta\right]$$

[$\because$ $\cos(\pi + x) = -\cos x$, $\cot(\pi + x) = \cot x$ and $\cot(2\pi + x) = \cot x$]

$$= \sin\theta\cos\theta[\tan\theta + \cot\theta] \quad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$

$$= (\sin\theta\cos\theta)\left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right] = (\sin\theta\cos\theta)\cdot\frac{(\sin^2\theta + \cos^2\theta)}{(\sin\theta\cos\theta)} = 1.$$

EXAMPLE 11 *Prove that*

(i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12} = \frac{\sqrt{3}}{2}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4} = \frac{-(\sqrt{3}+1)}{2\sqrt{2}}$

SOLUTION (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$

$$= \sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right) \quad [\because \sin x\cos y - \cos x\sin y = \sin(x-y)]$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) \quad [\because \sin x\cos y + \cos x\sin y = \sin(x+y)]$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

(iii) $\sin\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

$$= \cos\left(\pi - \frac{\pi}{3}\right)\cos\frac{\pi}{4} - \sin\left(\pi - \frac{\pi}{3}\right)\sin\frac{\pi}{4}$$

$$= -\cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\cos\frac{\pi}{4}$$

$$[\because \sin(\pi - x) = \sin x,\ \cos(\pi - x) = -\cos x]$$

$$= \left(-\frac{1}{2}\times\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right) = -\left\{\frac{(\sqrt{3}+1)}{2\sqrt{2}}\right\}.$$

EXAMPLE 12 *Evaluate: (i)* $\sin\frac{\pi}{12}$ *(ii)* $\sin\frac{5\pi}{12}$

SOLUTION (i) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

(ii) $\sin \dfrac{5\pi}{12} = \sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right)$

$= \sin \dfrac{\pi}{4} \cos \dfrac{\pi}{6} + \cos \dfrac{\pi}{4} \sin \dfrac{\pi}{6}$

$= \left(\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{1}{\sqrt{2}} \times \dfrac{1}{2}\right) = \dfrac{(\sqrt{3}+1)}{2\sqrt{2}}.$

EXAMPLE 13 *Prove that* $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

SOLUTION We have

$$\text{LHS} = \frac{\sin(x+y)}{\sin(x-y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{\tan x + \tan y}{\tan x - \tan y} \quad \text{[dividing num. and denom. by } \cos x \cos y\text{]}$$

$= \text{RHS}.$

EXAMPLE 14 *If* $\sin\theta = \dfrac{15}{17}$ *and* $\cos\phi = \dfrac{12}{13}$, *where* θ *and* ϕ *both lie in the first quadrant, find the values of*

(i) $\sin(\theta+\phi)$, *(ii)* $\cos(\theta-\phi)$, *(iii)* $\tan(\theta+\phi)$.

SOLUTION Given: $\sin\theta = \dfrac{15}{17}$ and $\cos\phi = \dfrac{12}{13}$.

Since θ lies in the first quadrant, we have

$\sin\theta > 0$, $\cos\theta > 0$ and $\tan\theta > 0$.

Again, since ϕ lies in the first quadrant, we have

$\sin\phi > 0$, $\cos\phi > 0$ and $\tan\phi > 0$.

Now, $\cos^2\theta = (1 - \sin^2\theta) = \left(1 - \dfrac{225}{289}\right) = \dfrac{64}{289}$

$\Rightarrow \quad \cos\theta = +\sqrt{\dfrac{64}{289}} = \dfrac{8}{17}.$

And, $\sin^2\phi = (1 - \cos^2\phi) = \left(1 - \dfrac{144}{169}\right) = \dfrac{25}{169}$

$\Rightarrow \quad \sin\phi = +\sqrt{\dfrac{25}{169}} = \dfrac{5}{13}.$

$\therefore \quad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \left(\dfrac{15}{17} \times \dfrac{17}{8}\right) = \dfrac{15}{8}$ and

$\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \left(\dfrac{5}{13} \times \dfrac{13}{12}\right) = \dfrac{5}{12}.$

Now, (i) $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= \left(\frac{15}{17}\times\frac{12}{13}\right) + \left(\frac{8}{17}\times\frac{5}{13}\right) = \left(\frac{180}{221}+\frac{40}{221}\right) = \frac{220}{221}.$$

(ii) $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= \left(\frac{8}{17}\times\frac{12}{13}\right) + \left(\frac{15}{17}\times\frac{5}{13}\right) = \left(\frac{96}{221}+\frac{75}{221}\right) = \frac{171}{221}.$$

(iii) $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$$= \frac{\left(\frac{15}{8}+\frac{5}{12}\right)}{\left\{1-\left(\frac{15}{8}\times\frac{5}{12}\right)\right\}} = \frac{\left(\frac{55}{24}\right)}{\left(1-\frac{25}{32}\right)} = \left(\frac{55}{24}\times\frac{32}{7}\right) = \frac{220}{21}.$$

EXAMPLE 15 *If* $\sin\theta = \frac{3}{5}$ *and* $\cos\phi = \frac{-12}{13}$, *where* θ *and* ϕ *both lie in the second quadrant, find the values of*

(i) $\sin(\theta-\phi)$, *(ii)* $\cos(\theta+\phi)$, *(iii)* $\tan(\theta-\phi)$.

SOLUTION Given: $\sin\theta = \frac{3}{5}$ and $\cos\phi = \frac{-12}{13}$.

Since θ lies in the second quadrant, we have

$\sin\theta > 0$, $\cos\theta < 0$ and $\tan\theta < 0$.

Again, since ϕ lies in the second quadrant, we have

$\sin\phi > 0$, $\cos\phi < 0$ and $\tan\phi < 0$.

Now, $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = \frac{-4}{5}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \frac{3}{5}\times\left(-\frac{5}{4}\right) = \frac{-3}{4}$.

Also, $\sin\phi = +\sqrt{1-\cos^2\phi} = +\sqrt{1-\frac{144}{169}} = +\sqrt{\frac{25}{169}} = +\frac{5}{13}$

and $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \frac{5}{13}\times\left(\frac{-13}{12}\right) = \frac{-5}{12}$.

$\therefore$ (i) $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$$= \left\{\frac{3}{5}\times\left(\frac{-12}{13}\right)\right\} - \left\{\left(\frac{-4}{5}\right)\times\frac{5}{13}\right\} = \left(\frac{-36}{65}+\frac{20}{65}\right) = \frac{-16}{65}.$$

(ii) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$= \left\{\left(\frac{-4}{5}\right)\times\left(\frac{-12}{13}\right)\right\} - \left\{\frac{3}{5}\times\frac{5}{13}\right\} = \left(\frac{48}{65}-\frac{3}{13}\right) = \frac{33}{65}.$$

(iii) $\tan(\theta-\phi) = \dfrac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

$$= \frac{\left(\frac{-3}{4}\right)-\left(\frac{-5}{12}\right)}{1+\left\{\left(\frac{-3}{4}\right)\times\left(\frac{-5}{12}\right)\right\}} = \frac{\left(\frac{-3}{4}+\frac{5}{12}\right)}{\left(1+\frac{5}{16}\right)} = \left(\frac{-1}{3}\right)\times\frac{16}{21} = \frac{-16}{63}.$$

EXAMPLE 16 *If* $\cos\theta = \frac{4}{5}$ *and* $\cos\phi = \frac{12}{13}$, *where* θ *and* ϕ *both lie in the fourth quadrant, find the values of*

(i) $\cos(\theta+\phi)$, *(ii)* $\sin(\theta-\phi)$, *(iii)* $\tan(\theta+\phi)$.

SOLUTION Given: $\cos\theta = \frac{4}{5}$ and $\cos\phi = \frac{12}{13}$.

Since θ lies in the fourth quadrant, we have

$\cos\theta > 0$, $\sin\theta < 0$ and $\tan\theta < 0$.

Again, since ϕ lies in the fourth quadrant, we have

$\cos\phi > 0$, $\sin\phi < 0$ and $\tan\phi < 0$.

Now, $\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = \frac{-3}{5}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \left(\frac{-3}{5}\right)\times\frac{5}{4} = \frac{-3}{4}$.

Also, $\sin\phi = -\sqrt{1-\cos^2\phi} = -\sqrt{1-\frac{144}{169}} = -\sqrt{\frac{25}{169}} = \frac{-5}{13}$

and $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \left(\frac{-5}{13}\right)\times\frac{13}{12} = \frac{-5}{12}$.

$\therefore$ (i) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$= \left(\frac{4}{5}\times\frac{12}{13}\right)-\left\{\left(\frac{-3}{5}\right)\times\left(\frac{-5}{13}\right)\right\} = \left(\frac{48}{65}-\frac{15}{65}\right) = \frac{33}{65}.$$

(ii) $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$$= \left\{\left(\frac{-3}{5}\right)\times\left(\frac{12}{13}\right)\right\}-\left\{\frac{4}{5}\times\left(\frac{-5}{13}\right)\right\} = \left(\frac{-36}{65}+\frac{20}{65}\right) = \frac{-16}{65}.$$

(iii) $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$$= \frac{\left(\frac{-3}{4}\right)+\left(\frac{-5}{12}\right)}{1-\left\{\left(\frac{-3}{4}\right)\times\left(\frac{-5}{12}\right)\right\}} = \frac{\left(\frac{-7}{6}\right)}{\left(1-\frac{5}{16}\right)} = \left(\frac{-7}{6}\right)\times\frac{16}{11} = \frac{-56}{33}.$$

EXAMPLE 17 *If* $\cot\alpha = \frac{1}{2}$ *and* $\sec\beta = \frac{-5}{3}$, *where* $\alpha \in \left]\pi, \frac{3\pi}{2}\right[$ *and* $\beta \in \left]\frac{\pi}{2}, \pi\right[$, *find the value of* $\tan(\alpha+\beta)$.

SOLUTION Here, α lies in Quadrant III and, therefore, tan α is positive.

Now, $\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2.$

Again, β lies in Quadrant II and, therefore, sin β is positive and cos β is negative.

Now, $\sec\beta = \frac{-5}{3} \Rightarrow \cos\beta = \frac{-3}{5}.$

And, $\sin\beta = +\sqrt{1-\cos^2\beta} = +\sqrt{1-\frac{9}{25}} = +\sqrt{\frac{16}{25}} = \frac{4}{5}.$

$$\therefore \quad \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{4}{5} \times \left(\frac{-5}{3}\right) = \frac{-4}{3}.$$

$$\therefore \quad \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{2 - \frac{4}{3}}{1 - 2\times\left(\frac{-4}{3}\right)} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{11}{3}\right)} = \left(\frac{2}{3}\times\frac{3}{11}\right) = \frac{2}{11}.$$

EXAMPLE 18 *Prove that* $\tan 56^\circ = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}.$

SOLUTION We have

$$\text{LHS} = \tan 56^\circ = \tan(45^\circ + 11^\circ)$$

$$= \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ}$$

$$= \frac{1 + \tan 11^\circ}{1 - \tan 11^\circ}$$

$$= \frac{1 + \frac{\sin 11^\circ}{\cos 11^\circ}}{1 - \frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \text{RHS}.$$

EXAMPLE 19 *Prove that* $\tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ.$

SOLUTION We have

$$\tan 70^\circ = \tan(20^\circ + 50^\circ)$$

$$\Rightarrow \quad \tan 70^\circ = \frac{\tan 20^\circ + \tan 50^\circ}{1 - \tan 20^\circ \tan 50^\circ}$$

$$\Rightarrow \quad \tan 70^\circ - \tan 70^\circ \tan 20^\circ \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$$

$$\Rightarrow \quad \tan 70^\circ - \tan(90^\circ - 20^\circ)\tan 20^\circ \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$$

$$\Rightarrow \quad \tan 70^\circ - \cot 20^\circ \tan 20^\circ \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$$

$$\Rightarrow \quad \tan 70^\circ - \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$$

$$\Rightarrow \quad \tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ.$$

EXAMPLE 20 *Show that* $\cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1.$

SOLUTION We have

$$\cot 3x = \cot(2x + x)$$

$$\Leftrightarrow \quad \cot 3x = \frac{\cot 2x \cot x - 1}{\cot 2x + \cot x}$$

$$\Leftrightarrow \quad \cot 3x \cot 2x + \cot 3x \cot x = \cot 2x \cot x - 1$$

$$\Leftrightarrow \quad \cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1.$$

EXERCISE 15B

1. Find the value of

(i) $\cos 840°$ (ii) $\sin 870°$ (iii) $\tan(-120°)$

(iv) $\sec(-420°)$ (v) $\operatorname{cosec}(-690°)$ (vi) $\tan(225°)$

(vii) $\cot(-315°)$ (viii) $\sin(-1230°)$ (ix) $\cos(495°)$

2. Find the values of all trigonometric functions of 135°.

3. Prove that

(i) $\sin 80° \cos 20° - \cos 80° \sin 20° = \frac{\sqrt{3}}{2}$

(ii) $\cos 45° \cos 15° - \sin 45° \sin 15° = \frac{1}{2}$

(iii) $\cos 75° \cos 15° + \sin 75° \sin 15° = \frac{1}{2}$

(iv) $\sin 40° \cos 20° + \cos 40° \sin 20° = \frac{\sqrt{3}}{2}$

(v) $\cos 130° \cos 40° + \sin 130° \sin 40° = 0$

4. Prove that

(i) $\sin(50° + \theta)\cos(20° + \theta) - \cos(50° + \theta)\sin(20° + \theta) = \frac{1}{2}$

(ii) $\cos(70° + \theta)\cos(10° + \theta) + \sin(70° + \theta)\sin(10° + \theta) = \frac{1}{2}$

5. Prove that

(i) $\cos(n+2)x \cos(n+1)x + \sin(n+2)x \sin(n+1)x = \cos x$

(ii) $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x + y)$

6. Prove that $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$.

7. Prove that

(i) $\sin 75^\circ=\dfrac{(\sqrt{6}+\sqrt{2})}{4}$

(ii) $\dfrac{\cos 135^\circ-\cos 120^\circ}{\cos 135^\circ+\cos 120^\circ}=(3-2\sqrt{2})$

(iii) $\tan 15^\circ+\cot 15^\circ=4$

8. Prove that

(i) $\cos 15^\circ-\sin 15^\circ=\dfrac{1}{\sqrt{2}}$

(ii) $\cot 105^\circ-\tan 105^\circ=2\sqrt{3}$

(iii) $\dfrac{\tan 69^\circ+\tan 66^\circ}{1-\tan 69^\circ\tan 66^\circ}=-1$

9. Prove that $\dfrac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\tan 54^\circ$.

10. Prove that $\dfrac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ}=\tan 37^\circ$.

11. Prove that $\dfrac{\cos(\pi+\theta)\cos(-\theta)}{\cos(\pi-\theta)\cos\left(\dfrac{\pi}{2}+\theta\right)}=-\cot\theta$.

12. Prove that $\dfrac{\cos\theta}{\sin(90^\circ+\theta)}+\dfrac{\sin(-\theta)}{\sin(180^\circ+\theta)}-\dfrac{\tan(90^\circ+\theta)}{\cot\theta}=3$.

13. Prove that $\dfrac{\sin(180^\circ+\theta)\cos(90^\circ+\theta)\tan(270^\circ-\theta)\cot(360^\circ-\theta)}{\sin(360^\circ-\theta)\cos(360^\circ+\theta)\operatorname{cosec}(-\theta)\sin(270^\circ+\theta)}=1$.

14. If θ and ϕ lie in the first quadrant such that $\sin\theta=\dfrac{8}{17}$ and $\cos\phi=\dfrac{12}{13}$, find the values of

(i) $\sin(\theta-\phi)$, (ii) $\cos(\theta+\phi)$, (iii) $\tan(\theta-\phi)$.

15. If x and y are acute angles such that $\sin x=\dfrac{1}{\sqrt{5}}$ and $\sin y=\dfrac{1}{\sqrt{10}}$, prove that $(x+y)=\dfrac{\pi}{4}$.

16. If x and y are acute angles such that $\cos x=\dfrac{13}{14}$ and $\cos y=\dfrac{1}{7}$, prove that $(x-y)=-\dfrac{\pi}{3}$.

17. If $\sin x = \frac{12}{13}$ and $\sin y = \frac{4}{5}$, where $\frac{\pi}{2} < x < \pi$ and $0 < y < \frac{\pi}{2}$, find the values of

(i) $\sin(x + y)$, (ii) $\cos(x + y)$, (iii) $\tan(x - y)$.

18. If $\cos x = \frac{3}{5}$ and $\cos y = \frac{-24}{25}$, where $\frac{3\pi}{2} < x < 2\pi$ and $\pi < y < \frac{3\pi}{2}$, find the values of

(i) $\sin(x + y)$, (ii) $\cos(x - y)$, (iii) $\tan(x + y)$.

19. Prove that

(i) $\cos\left(\frac{\pi}{3} + x\right) = \frac{1}{2}(\cos x - \sqrt{3}\sin x)$

(ii) $\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$

(iii) $\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4} + x\right) = \frac{1}{2}(\cos x - \sin x)$

(iv) $\cos x + \cos\left(\frac{2\pi}{3} + x\right) + \cos\left(\frac{2\pi}{3} - x\right) = 0$

20. Prove that

(i) $2\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \frac{1}{2}$

(ii) $2\cos\frac{5\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}$

(iii) $2\sin\frac{5\pi}{12}\cos\frac{\pi}{12} = \frac{(2+\sqrt{3})}{2}$

ANSWERS (EXERCISE 15B)

1. (i) $-\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\sqrt{3}$ (iv) 2 (v) 2 (vi) 1 (vii) 1 (viii) $-\frac{1}{2}$ (ix) $-\frac{1}{\sqrt{2}}$

2. $\sin 135° = \frac{1}{\sqrt{2}}$, $\cos 135° = \frac{-1}{\sqrt{2}}$, $\tan 135° = -1$, $\cot 135° = -1$,

$\sec 135° = -\sqrt{2}$, $\operatorname{cosec} 135° = \sqrt{2}$

14. (i) $\frac{21}{221}$ (ii) $\frac{140}{221}$ (iii) $\frac{21}{220}$

17. (i) $\frac{16}{65}$ (ii) $\frac{-63}{65}$ (iii) $\frac{56}{33}$

18. (i) $\frac{3}{5}$ (ii) $\frac{-44}{125}$ (iii) $\frac{-3}{4}$

4. (i) LHS $= \sin x \cos y - \cos x \sin y$

$= \sin(x - y) = \sin[(50^\circ + \theta) - (20^\circ + \theta)] = \sin 30^\circ.$

(ii) LHS $= \cos x \cos y + \sin x \sin y$

$= \cos(x - y) = \cos[(70^\circ + \theta) - (10^\circ + \theta)] = \cos 60^\circ.$

5. (i) LHS $= \cos A \cos B + \sin A \sin B$

$= \cos(A - B) = \cos[(n + 2)x - (n + 1)x] = \cos x.$

(ii) LHS $= \cos A \cos B - \sin A \sin B$

$$= \cos(A + B) = \cos\left[\left(\frac{\pi}{4} - x\right) + \left(\frac{\pi}{4} - y\right)\right]$$

$$= \cos\left[\frac{\pi}{2} - (x + y)\right] = \sin(x + y).$$

7. (i) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

(ii) $\cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45^\circ.$

$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ.$

(iii) $\tan 15^\circ = \tan(45^\circ - 30^\circ) = \dfrac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}.$

8. (i) $\cos 15^\circ = \cos(45^\circ - 30^\circ)$ and $\sin 15^\circ = \sin(45^\circ - 30^\circ)$.

(ii) $\tan 105^\circ = \tan(60^\circ + 45^\circ)$.

(iii) LHS $= \tan(69^\circ + 66^\circ) = \tan 135^\circ$

$= \tan(180^\circ - 45^\circ) = -\tan 45^\circ = -1.$

9. Dividing num. and denom. by $\cos 9^\circ$, we get

$$\text{LHS} = \frac{1 + \tan 9^\circ}{1 - \tan 9^\circ} = \frac{\tan 45^\circ + \tan 9^\circ}{1 - \tan 45^\circ \tan 9^\circ}$$

$$= \tan(45^\circ + 9^\circ) = \tan 54^\circ.$$

17. Clearly, $\sin x > 0$, $\cos x < 0$ and $\tan x < 0$.

Also, $\sin y > 0$, $\cos y > 0$ and $\tan y > 0$.

18. In the 4th quadrant, $\cos x > 0$, $\sin x < 0$ and $\tan x < 0$.

In the 3rd quadrant, $\tan y > 0$, $\sin y < 0$ and $\cos y < 0$.

$$\therefore \quad \sin x = -\sqrt{1 - \frac{9}{25}} = -\sqrt{\frac{16}{25}} = \frac{-4}{5},$$

$$\tan x = \left(\frac{-4}{5} \times \frac{5}{3}\right) = \frac{-4}{3}$$

$$\cos y = \frac{-24}{25}, \ \sin y = -\sqrt{1 - \frac{576}{625}} = -\sqrt{\frac{49}{625}} = \frac{-7}{25} \text{ and } \tan y = \frac{7}{24}.$$

Now, find the required values.

SOME MORE TRIGONOMETRIC FUNCTIONS

THEOREM 1 *For all* $x, y \in R$, *we have*

(a) $2\sin x \cos y = \sin(x+y) + \sin(x-y)$

(b) $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

(c) $2\cos x \cos y = \cos(x+y) + \cos(x-y)$

(d) $2\sin x \sin y = \cos(x-y) - \cos(x+y)$

PROOF We know that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \ldots \text{(i)}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \quad \ldots \text{(ii)}$$

(a) Adding (i) and (ii), we get

$$2\sin x \cos y = \sin(x+y) + \sin(x-y).$$

(b) Subtracting (ii) from (i), we get

$$2\cos x \sin y = \sin(x+y) - \sin(x-y).$$

Further, we know that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad \ldots \text{(iii)}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad \ldots \text{(iv)}$$

(c) Adding (iii) and (iv), we get

$$2\cos x \cos y = \cos(x+y) + \cos(x-y).$$

(d) Subtracting (iii) from (iv), we get

$$2\sin x \sin y = \cos(x-y) - \cos(x+y).$$

THEOREM 2 *For all* $x, y \in R$, *we have*

(i) $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(ii) $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

(iii) $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(iv) $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

PROOF Putting $x = (a+b)$ and $y = (a-b)$, we get $a = \left(\frac{x+y}{2}\right)$ and $b = \left(\frac{x-y}{2}\right)$.

(i) $\sin x + \sin y = \sin(a+b) + \sin(a-b)$

$$= 2\sin a \cos b$$

$$= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

(ii) $\sin x - \sin y = \sin(a+b) - \sin(a-b)$

$$= 2\cos a \sin b$$

$$= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

(iii) $\cos x + \cos y = \cos(a+b) + \cos(a-b)$

$= 2\cos a \cos b$

$= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

(iv) $\cos x - \cos y = \cos(a+b) - \cos(a-b)$

$= -2\sin a \sin b$

$= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

These formulae may be listed as under:

I. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

II. $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

III. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

IV. $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

SOLVED EXAMPLES

EXAMPLE 1 *Prove that*

(i) $\cos\left(\frac{\pi}{4}+x\right) + \cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$

(ii) $\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

SOLUTION (i) LHS $= \cos\left(\frac{\pi}{4}+x\right) + \cos\left(\frac{\pi}{4}-x\right)$

$= 2\cos\frac{\pi}{4}\cos x$ [$\because \cos(A+B) + \cos(A-B) = 2\cos A\cos B$]

$= \left(2\times\frac{1}{\sqrt{2}}\cos x\right) = \sqrt{2}\cos x =$ RHS.

(ii) LHS $= \cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right)$

$= -2\sin\frac{3\pi}{4}\sin x$

[$\because \cos(A+B) - \cos(A-B) = -2\sin A\sin B$]

$= -2\sin\left(\pi - \frac{\pi}{4}\right)\sin x$

$$= -2\sin\frac{\pi}{4}\sin x \qquad \left[\because \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4}\right]$$

$$= \left(-2 \times \frac{1}{\sqrt{2}}\right)\sin x = -\sqrt{2}\sin x = \text{RHS.}$$

EXAMPLE 2 *Express each of the following as an algebraic sum of sines or cosines:*

(i) $2\sin 3x \cos 2x$ *(ii)* $2\cos 4x \sin 2x$

(iii) $2\cos 6x \cos 4x$ *(iv)* $2\sin 3x \sin 5x$

SOLUTION (i) We know that $2\sin A \cos B = \sin(A + B) + \sin(A - B)$.

$$\therefore \quad 2\sin 3x \cos 2x = \sin(3x + 2x) + \sin(3x - 2x)$$
$$= \sin 5x + \sin x.$$

(ii) We know that $2\cos A \sin B = \sin(A + B) - \sin(A - B)$.

$$\therefore \quad 2\cos 4x \sin 2x = \sin(4x + 2x) - \sin(4x - 2x)$$
$$= \sin 6x - \sin 2x.$$

(iii) We know that $2\cos A \cos B = \cos(A + B) + \cos(A - B)$.

$$\therefore \quad 2\cos 6x \cos 4x = \cos(6x + 4x) + \cos(6x - 4x)$$
$$= \cos 10x + \cos 2x.$$

(iv) We know that $2\sin A \sin B = \cos(A - B) - \cos(A + B)$.

$$\therefore \quad 2\sin 3x \sin 5x = 2\sin 5x \sin 3x$$
$$= \cos(5x - 3x) - \cos(5x + 3x)$$
$$= \cos 2x - \cos 8x.$$

EXAMPLE 3 *Express each of the following as a product of sines or cosines or sine and cosine:*

(i) $\cos 5x + \cos 3x$ *(ii)* $\cos 5x - \cos 7x$

(iii) $\sin 7x + \sin 3x$ *(iv)* $\sin 5x - \sin 3x$

SOLUTION (i) We know that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \cos 5x + \cos 3x = 2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$
$$= 2\cos 4x \cos x.$$

(ii) We know that $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \cos 5x - \cos 7x = -2\sin\left(\frac{5x+7x}{2}\right)\sin\left(\frac{5x-7x}{2}\right)$$
$$= -2\sin 6x \sin(-x) = 2\sin 6x \sin x.$$

(iii) We know that $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \sin 7x + \sin 3x = 2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)$$
$$= 2\sin 5x \cos 2x.$$

(iv) We know that $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \sin 5x - \sin 3x = 2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)$$

$$= 2\cos 4x \sin x.$$

EXAMPLE 4 *Prove that* $\frac{\cos 6x + \cos 4x}{\sin 6x - \sin 4x} = \cot x.$

SOLUTION We have

$$\text{LHS} = \frac{\cos 6x + \cos 4x}{\sin 6x - \sin 4x}$$

$$= \frac{2\cos\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right)}{2\cos\left(\frac{6x+4x}{2}\right)\sin\left(\frac{6x-4x}{2}\right)}$$

$$\left[\begin{array}{l}\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \\ \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\end{array}\right]$$

$$= \frac{2\cos 5x \cos x}{2\cos 5x \sin x} = \frac{\cos x}{\sin x}$$

$$= \cot x = \text{RHS}.$$

EXAMPLE 5 *Prove that* $\frac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x.$

SOLUTION We have

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos x - \cos 3x}$$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$$

$$\left[\begin{array}{l}\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\end{array}\right]$$

$$= \frac{\cos 2x \sin x}{-\sin 2x \sin(-x)} = \frac{\cos 2x \sin x}{\sin 2x \sin x}$$

$$= \cot 2x = \text{RHS}.$$

EXAMPLE 6 *Prove that* $\dfrac{\sin 3x - \sin x}{\cos 2x} = 2\sin x.$

SOLUTION We have

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos 2x}$$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{\cos 2x}$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \frac{2\cos 2x \sin x}{\cos 2x} = 2\sin x = \text{RHS.}$$

EXAMPLE 7 *Prove that* $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \operatorname{cosec} 2x - \cot 2x.$

SOLUTION We have

$$\text{LHS} = \frac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)}$$

$$\left[\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \text{ and } \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \frac{2\sin 3x \cos 2x - 2\sin 3x}{-2\sin 3x \sin 2x} = \frac{2\sin 3x(\cos 2x - 1)}{-2\sin 3x \sin 2x}$$

$$= \frac{(1-\cos 2x)}{\sin 2x} = \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}$$

$$= \operatorname{cosec} 2x - \cot 2x = \text{RHS.}$$

EXAMPLE 8 *If* $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\sin(C+D)}{\sin(C-D)}$, *prove that* $\tan A \tan B \tan C + \tan D = 0.$

SOLUTION We have

$$\frac{\cos(A+B)}{\cos(A-B)} = \frac{\sin(C+D)}{\sin(C-D)}$$

$$\Rightarrow \frac{\cos(A+B) + \cos(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{\sin(C+D) + \sin(C-D)}{\sin(C+D) - \sin(C-D)}$$

$$\Rightarrow \quad \frac{2\cos A\ \cos B}{-2\sin A\ \sin B} = \frac{2\sin C\ \cos D}{2\cos C\ \sin D}$$

$\Rightarrow \quad -\cot A\ \cot B = \tan C\ \cot D$

$\Rightarrow \quad \tan A\ \tan B\ \tan C = -\tan D$

$\Rightarrow \quad \tan A\ \tan B\ \tan C + \tan D = 0.$

Hence, $\tan A\ \tan B\ \tan C + \tan D = 0.$

EXAMPLE 9 *Prove that* $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x.$

SOLUTION We have

$$\text{LHS} = \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$= \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \sin 3x}$$

[using formulae for $(\cos C + \cos D)$ and $(\sin C + \sin D)$]

$$= \frac{2\cos 3x\ \cos x + \cos 3x}{2\sin 3x\ \cos x + \sin 3x} = \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)}$$

$$= \frac{\cos 3x}{\sin 3x} = \cot 3x = \text{RHS}.$$

EXAMPLE 10 *Prove that* $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\sin 4x\ \cos 2x\ \cos x.$

SOLUTION We have

$$\text{LHS} = (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$$

$$= 2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= 2\sin 4x\ \cos 3x + 2\sin 4x\ \cos x$$

$$= 2\sin 4x\ (\cos 3x + \cos x)$$

$$= (2\sin 4x) \times 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$= (2\sin 4x) \times 2\cos 2x\ \cos x$$

$$= 4\sin 4x\ \cos 2x\ \cos x = \text{RHS}.$$

EXAMPLE 11 *Prove that* $\cos 2x\ \cos \dfrac{x}{2} - \cos 3x\ \cos \dfrac{9x}{2} = -\sin 5x\ \sin\left(\dfrac{5x}{2}\right).$

SOLUTION We have

$$\text{LHS} = \frac{1}{2}\left[2\cos 2x\ \cos\frac{x}{2} - 2\cos 3x\ \cos\frac{9x}{2}\right]$$

$$= \frac{1}{2}\left[\cos\left(2x + \frac{x}{2}\right) + \cos\left(2x - \frac{x}{2}\right)\right] - \frac{1}{2}\left[\cos\left(\frac{9x}{2} + 3x\right) + \cos\left(\frac{9x}{2} - 3x\right)\right]$$

$$= \frac{1}{2}\left(\cos\frac{5x}{2} + \cos\frac{3x}{2}\right) - \frac{1}{2}\left(\cos\frac{15x}{2} + \cos\frac{3x}{2}\right)$$

$$= \frac{1}{2}\left(\cos\frac{5x}{2} - \cos\frac{15x}{2}\right)$$

$$= \frac{1}{2}\left[-2\sin\frac{\left(\frac{5x}{2}+\frac{15x}{2}\right)}{2}\sin\frac{\left(\frac{15x}{2}-\frac{5x}{2}\right)}{2}\right]$$

$$= -\sin 5x \sin\left(\frac{5x}{2}\right) = \text{RHS.}$$

EXAMPLE 12 *Prove that* $\dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 4x \sin 3x} = \tan 2x.$

SOLUTION We have

$$\text{LHS} = \frac{2\sin 8x \cos x - 2\sin 6x \cos 3x}{2\cos 2x \cos x - 2\sin 4x \sin 3x}$$

$$= \frac{[\sin(8x+x) + \sin(8x-x)] - [\sin(6x+3x) + \sin(6x-3x)]}{[\cos(2x+x) + \cos(2x-x)] - [\cos(4x-3x) - \cos(4x+3x)]}$$

$$[\because\ 2\sin A \cos B = \sin(A+B) + \sin(A-B), \text{ etc.}]$$

$$= \frac{(\sin 9x + \sin 7x) - (\sin 9x + \sin 3x)}{(\cos 3x + \cos x) - (\cos x - \cos 7x)}$$

$$= \frac{(\sin 7x - \sin 3x)}{(\cos 3x + \cos 7x)}$$

$$= \frac{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}{2\cos\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)}$$

$$\left[\because\ \sin C - \sin D = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}\right]$$

$$= \frac{\cos 5x \sin 2x}{\cos 5x \cos 2x} = \frac{\sin 2x}{\cos 2x}$$

$$= \tan 2x = \text{RHS.}$$

EXAMPLE 13 *Prove that* $\dfrac{(\sin x - \sin y)}{(\cos x + \cos y)} = \tan\left(\dfrac{x-y}{2}\right).$

SOLUTION We have

$$\text{LHS} = \frac{\sin x - \sin y}{\cos x + \cos y}$$

$$= \frac{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \tan\left(\frac{x-y}{2}\right) = \text{RHS.}$$

EXAMPLE 14 *Prove that* $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4\cos^2\left(\frac{x-y}{2}\right)$.

SOLUTION We have

$$\text{LHS} = (\cos x + \cos y)^2 + (\sin x + \sin y)^2$$

$$= \left\{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right\}^2$$

$$= 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)\cdot\left\{\cos^2\left(\frac{x+y}{2}\right) + \sin^2\left(\frac{x+y}{2}\right)\right\}$$

$$= 4\cos^2\left(\frac{x-y}{2}\right) = \text{RHS}.$$

EXAMPLE 15 *Prove that*

$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$$
$$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right).$$

SOLUTION We have

$$\text{LHS} = (\cos\alpha + \cos\beta) + \{\cos(\alpha+\beta+\gamma) + \cos\gamma\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\left\{2\cos\left(\frac{\gamma+\alpha}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

$$\left[\text{using } \cos C + \cos D = 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2}\right]$$

$$= 4\cos\left(\frac{\alpha+\beta}{2}\right)2\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{RHS}.$$

EXAMPLE 16 *Prove that*

$$\sin^2 6x - \sin^2 4x = \sin 10x \sin 2x.$$

SOLUTION $\text{LHS} = \sin^2 6x - \sin^2 4x$

$= \sin(6x + 4x)\sin(6x - 4x)$

$[\because \sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)]$

$= \sin 10x \sin 2x = \text{RHS}.$

EXAMPLE 17 *Prove that* $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$.

SOLUTION We have

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$$
$$= \frac{1}{2}\cos 60^\circ \cos 40^\circ (2\cos 80^\circ \cos 20^\circ)$$
$$= \frac{1}{2} \times \frac{1}{2}\cos 40^\circ \cdot \{\cos(80^\circ + 20^\circ) + \cos(80^\circ - 20^\circ)\}$$
$$= \frac{1}{4}\cos 40^\circ (\cos 100^\circ + \cos 60^\circ)$$
$$= \frac{1}{4}\cos 40^\circ \left(\cos 100^\circ + \frac{1}{2}\right)$$
$$= \frac{1}{8}(2\cos 100^\circ \cos 40^\circ) + \frac{1}{8}\cos 40^\circ$$
$$= \frac{1}{8}[\cos(100^\circ + 40^\circ) + \cos(100^\circ - 40^\circ)] + \frac{1}{8}\cos 40^\circ$$
$$= \frac{1}{8}(\cos 140^\circ + \cos 60^\circ) + \frac{1}{8}\cos 40^\circ$$
$$= \frac{1}{8}\cos 140^\circ + \frac{1}{16} + \frac{1}{8}\cos 40^\circ$$
$$= \frac{1}{8}\cos(180^\circ - 40^\circ) + \frac{1}{16} + \frac{1}{8}\cos 40^\circ$$
$$= -\frac{1}{8}\cos 40^\circ + \frac{1}{16} + \frac{1}{8}\cos 40^\circ$$
$$= \frac{1}{16} = \text{RHS.}$$

EXAMPLE 18 *Prove that* $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \dfrac{\sqrt{3}}{16}$.

SOLUTION We have

$$\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ$$
$$= \frac{1}{2}\sin 60^\circ \sin 10^\circ (2\sin 70^\circ \sin 50^\circ)$$
$$= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \sin 10^\circ \cdot \{\cos(70^\circ - 50^\circ) - \cos(70^\circ + 50^\circ)\}$$
$$= \frac{\sqrt{3}}{4}\sin 10^\circ \cdot \{\cos 20^\circ - \cos 120^\circ\}$$
$$= \frac{\sqrt{3}}{4}\sin 10^\circ \left(\cos 20^\circ + \frac{1}{2}\right) \qquad \left[\because \cos 120^\circ = -\frac{1}{2}\right]$$
$$= \frac{\sqrt{3}}{8}(2\cos 20^\circ \sin 10^\circ) + \frac{\sqrt{3}}{8}\sin 10^\circ$$

$$= \frac{\sqrt{3}}{8}[\sin(20^\circ + 10^\circ) - \sin(20^\circ - 10^\circ)] + \frac{\sqrt{3}}{8}\sin 10^\circ$$

$$= \frac{\sqrt{3}}{8}\sin 30^\circ = \left(\frac{\sqrt{3}}{8} \times \frac{1}{2}\right) = \frac{\sqrt{3}}{16}.$$

EXERCISE 15C

Prove that

1. $\sin(150^\circ + x) + \sin(150^\circ - x) = \cos x$
2. $\cos x + \cos(120^\circ - x) + \cos(120^\circ + x) = 0$
3. $\sin\left(x - \frac{\pi}{6}\right) + \cos\left(x - \frac{\pi}{3}\right) = \sqrt{3}\sin x$
4. $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$
5. $\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$
6. Express each of the following as a product:
   (i) $\sin 10x + \sin 6x$ (ii) $\sin 7x - \sin 3x$
   (iii) $\cos 7x + \cos 5x$ (iv) $\cos 2x - \cos 4x$
7. Express each of the following as an algebraic sum of sines or cosines:
   (i) $2\sin 6x \cos 4x$ (ii) $2\cos 5x \sin 3x$
   (iii) $2\cos 7x \cos 3x$ (iv) $2\sin 8x \sin 2x$

Prove that

8. $\frac{\sin x + \sin 3x}{\cos x - \cos 3x} = \cot x$
9. $\frac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \tan x$
10. $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$
11. $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = \frac{-\sin 2x}{\cos 10x}$
12. $\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$
13. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$
14. $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$
15. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$
16. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\left(\frac{x - y}{2}\right)$

17. $\dfrac{\sin 2x - \sin 2y}{\cos 2y - \cos 2x} = \cot (x + y)$

18. $\dfrac{\cos x + \cos y}{\cos y - \cos x} = \cot\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{x-y}{2}\right)$

19. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \tan\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{x-y}{2}\right)$

20. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos\dfrac{x}{2}\cos\dfrac{3x}{2}$

21. $\dfrac{\cos 4x \sin 3x - \cos 2x \sin x}{\sin 4x \sin x + \cos 6x \cos x} = \tan 2x$

22. $\dfrac{\cos 2x \sin x + \cos 6x \sin 3x}{\sin 2x \sin x + \sin 6x \sin 3x} = \cot 5x$

23. $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \dfrac{1}{16}$

24. $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$

25. $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \dfrac{3}{16}$

26. If $\cos x + \cos y = \dfrac{1}{3}$ and $\sin x + \sin y = \dfrac{1}{4}$, prove that $\tan\left(\dfrac{x+y}{2}\right) = \dfrac{3}{4}$.

27. Prove that

(i) $2\cos 45^\circ \cos 15^\circ = \dfrac{(\sqrt{3}+1)}{2}$ (ii) $2\sin 75^\circ \sin 15^\circ = \dfrac{1}{2}$

(iii) $\cos 15^\circ - \sin 15 = \dfrac{1}{\sqrt{2}}$

ANSWERS (EXERCISE 15C)

6. (i) $2\sin 8x \cos 2x$ (ii) $2\cos 5x \sin 2x$ (iii) $2\cos 6x \cos x$ (iv) $2\sin 3x \sin x$

7. (i) $\sin 10x + \sin 2x$ (ii) $\sin 8x - \sin 2x$ (iii) $\cos 10x + \cos 4x$
(iv) $\cos 6x - \cos 10x$

HINTS TO SOME SELECTED QUESTIONS

16. LHS $= \left\{-2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)\right\}^2 + \left\{\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)\right\}^2$

$= 4\sin^2\left(\dfrac{x-y}{2}\right)\cdot\left\{\sin^2\left(\dfrac{x+y}{2}\right) + \cos^2\left(\dfrac{x+y}{2}\right)\right\} = 4\sin^2\left(\dfrac{x-y}{2}\right)$

21. LHS $= \dfrac{2\cos 4x \sin 3x - 2\cos 2x \sin x}{2\sin 4x \sin x + 2\cos 6x \cos x}$

$= \dfrac{\{\sin(4x+3x) - \sin(4x-3x)\} - \{\sin(2x+x) - \sin(2x-x)\}}{\{\cos(4x-x) - \cos(4x+x)\} + \{\cos(6x+x) + \cos(6x-x)\}}$

$= \dfrac{(\sin 7x - \sin x) - (\sin 3x - \sin x)}{(\cos 3x - \cos 5x) + (\cos 7x + \cos 5x)} = \dfrac{(\sin 7x - \sin 3x)}{(\cos 7x + \cos 3x)}$

$= \dfrac{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}{2\cos\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)} = \dfrac{\sin 2x}{\cos 2x} = \tan 2x = \text{RHS.}$

26. $\cos x + \cos y = \frac{1}{3} \Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{3}$ ... (i)

$\sin x + \sin y = \frac{1}{4} \Rightarrow 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{4}$ ... (ii)

Dividing (ii) by (i), we get $\tan\left(\frac{x+y}{2}\right) = \frac{3}{4}$.

27. (i) Use $2\cos x \cos y = \cos(x+y) + \cos(x-y)$.

(ii) Use $2\sin x \sin y = \cos(x-y) - \cos(x+y)$.

(iii) LHS $= \cos 15^\circ - \cos(90^\circ - 15^\circ) = \cos 15^\circ - \cos 75^\circ$.

Now, use $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

THEOREM 1 *(i)* $\sin 2x = 2\sin x \cos x$

(ii) $\cos 2x = (\cos^2 x - \sin^2 x) = (2\cos^2 x - 1) = (1 - 2\sin^2 x)$

(iii) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

PROOF (i) $\sin 2x = \sin(x+x)$

$= \sin x \cos x + \cos x \sin x$

$= 2\sin x \cos x.$

(ii) $\cos 2x = \cos(x+x)$

$= \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x.$

Now, $\cos 2x = \cos^2 x - \sin^2 x$

$= (1 - \sin^2 x - \sin^2 x) = (1 - 2\sin^2 x)$

$[\because \cos^2 x = (1 - \sin^2 x)].$

Also, $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$ $[\because \sin^2 x = 1 - \cos^2 x]$

$= (2\cos^2 x - 1).$

$\therefore \cos 2x = (\cos^2 x - \sin^2 x) = (1 - 2\sin^2 x) = (2\cos^2 x - 1).$

(iii) $\tan 2x = \tan(x + x)$

$$= \frac{\tan x + \tan x}{1 - (\tan x \times \tan x)}$$

$$= \frac{2\tan x}{1 - \tan^2 x}.$$

$$\therefore \quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

REMARK The above results may be expressed as

(i) $(1 - \cos 2x) = 2\sin^2 x$

(ii) $(1 + \cos 2x) = 2\cos^2 x$

(iii) $\dfrac{(1 - \cos 2x)}{(1 + \cos 2x)} = \tan^2 x$

THEOREM 2 *If x is not an odd multiple of $\frac{\pi}{2}$ then prove that*

(i) $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$ *(ii)* $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

PROOF Since x is not an odd multiple of $\frac{\pi}{2}$, we have $\cos x \neq 0$.

(i) $\sin 2x = 2\sin x \cos x$

$$= \frac{2\sin x \cos x}{\cos^2 x + \sin^2 x} \qquad [\because 1 = \cos^2 x + \sin^2 x]$$

$$= \frac{\left(\dfrac{2\sin x}{\cos x}\right)}{1 + \tan^2 x} \qquad \text{[dividing num. and denom. by } \cos^2 x\text{]}$$

$$= \frac{2\tan x}{1 + \tan^2 x}.$$

$$\therefore \quad \sin 2x = \frac{2\tan x}{1 + \tan^2 x}.$$

(ii) $\cos 2x = \cos^2 x - \sin^2 x$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \qquad [\because 1 = \cos^2 x + \sin^2 x]$$

$$= \frac{1 - \dfrac{\sin^2 x}{\cos^2 x}}{1 + \dfrac{\sin^2 x}{\cos^2 x}} \qquad \text{[dividing num. and denom. by } \cos^2 x\text{]}$$

$$= \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

$$\therefore \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

SUMMARY

(i) $\sin 2x = 2\sin x \cos x$

(ii) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(iii) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

(iv) $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$

(v) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

THEOREM 3 *(i)* $\sin 3x = 3\sin x - 4\sin^3 x$ *(ii)* $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

PROOF (i) $\sin 3x = \sin (2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin x \cos^2 x + \cos 2x \sin x$ $[\because \sin 2x = 2\sin x \cos x]$

$= 2\sin x(1 - \sin^2 x) + (1 - 2\sin^2 x) \sin x$

$[\because \cos^2 x = (1 - \sin^2 x)$ and $\cos 2x = (1 - 2\sin^2 x)]$

$= 3\sin x - 4\sin^3 x.$

(ii) $\cos 3x = \cos (2x + x)$

$= \cos 2x \cos x - \sin 2x \sin x$

$= (2\cos^2 x - 1) \cos x - 2\sin^2 x \cos x$

$[\because \cos 2x = (2\cos^2 x - 1)$ and $\sin 2x = 2\sin x \cos x]$

$= (2\cos^2 x - 1) \cos x + 2(\cos^2 x - 1) \cos x$

$[\because \sin^2 x = (1 - \cos^2 x)]$

$= 4\cos^3 x - 3\cos x.$

(iii) $\tan 3x = \tan (2x + x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$$= \frac{\dfrac{2\tan x}{(1 - \tan^2 x)} + \tan x}{1 - \left(\dfrac{2\tan x}{1 - \tan^2 x}\right) \cdot \tan x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

SUMMARY

(i) $\sin 3x = 3\sin x - 4\sin^3 x$

(ii) $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\sin x = -\frac{1}{2}$ *and* $\pi < x < \frac{3\pi}{2}$, *find the values of*

(i) $\sin 2x$, *(ii)* $\cos 2x$, *(iii)* $\tan 2x$.

SOLUTION Since x lies in Quadrant III, we have $\sin x < 0$, $\cos x < 0$ and $\tan x > 0$.

Now, $\sin x = -\frac{1}{2}$ (given).

$$\therefore \quad \cos x = -\sqrt{1-\sin^2 x} = -\sqrt{\left(1-\frac{1}{4}\right)} = -\sqrt{\frac{3}{4}} = \frac{-\sqrt{3}}{2}$$

$$\text{and } \tan x = \frac{\sin x}{\cos x} = \left(-\frac{1}{2} \times \frac{2}{-\sqrt{3}}\right) = \frac{1}{\sqrt{3}}.$$

(i) $\sin 2x = 2\sin x \cos x$

$$= 2 \times \left(-\frac{1}{2}\right) \times \left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

(ii) $\cos 2x = (2\cos^2 x - 1)$

$$= \left(2 \times \frac{3}{4}\right) - 1 = \left(\frac{3}{2} - 1\right) = \frac{1}{2}.$$

(iii) $$\tan 2x = \frac{\sin 2x}{\cos 2x} = \left(\frac{\sqrt{3}}{2} \times \frac{2}{1}\right) = \sqrt{3}.$$

EXAMPLE 2 *If* $\sec x = -\frac{13}{12}$ *and* $\frac{\pi}{2} < x < \pi$, *find the values of*

(i) $\sin 2x$, *(ii)* $\cos 2x$, *(iii)* $\tan 2x$.

SOLUTION Since x lies in Quadrant II, we have $\sin x > 0$, $\cos x < 0$ and $\tan x < 0$.

Now, $\sec x = \frac{-13}{12} \Rightarrow \cos x = \frac{1}{\sec x} = \frac{-12}{13}$.

$$\therefore \quad \sin x = +\sqrt{1-\cos^2 x} = +\sqrt{1-\frac{144}{169}} = +\sqrt{\frac{25}{169}} = \frac{5}{13}.$$

$$\text{And, } \tan x = \frac{\sin x}{\cos x} = \frac{5}{13} \times \left(\frac{-13}{12}\right) = \frac{-5}{12}.$$

(i) $\sin 2x = 2\sin x \cos x$

$$= \left\{2 \times \frac{5}{13} \times \frac{(-12)}{13}\right\} = \frac{-120}{169}.$$

(ii) $\cos 2x = 2\cos^2 x - 1$

$$= \left(2 \times \frac{144}{169} - 1\right) = \frac{119}{169}.$$

(iii) $$\tan 2x = \frac{\sin 2x}{\cos 2x} = \left(\frac{-120}{169} \times \frac{169}{119}\right) = \frac{-120}{119}.$$

EXAMPLE 3 *If* $\tan x = \frac{-3}{4}$ *and* $\frac{3\pi}{2} < x < 2\pi$, *find the values of*

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

SOLUTION Since x lies in Quadrant IV, we have

$$\cos x > 0,\ \sin x < 0 \text{ and } \tan x < 0.$$

$$\therefore \quad \sec x = \sqrt{1+\tan^2 x} = \sqrt{\left(1+\frac{9}{16}\right)} = \sqrt{\frac{25}{16}} = \frac{5}{4} \Rightarrow \cos x = \frac{4}{5}.$$

$$\text{And, } \sin x = -\sqrt{1-\cos^2 x} = -\sqrt{\left(1-\frac{16}{25}\right)} = -\sqrt{\frac{9}{25}} = \frac{-3}{5}.$$

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} = \left(\frac{-3}{5}\times\frac{5}{4}\right) = \frac{-3}{4}.$$

(i) $\sin 2x = 2\sin x \cos x$

$$= \left\{2\times\frac{(-3)}{5}\times\frac{4}{5}\right\} = \frac{-24}{25}.$$

(ii) $\cos 2x = (2\cos^2 x - 1)$

$$= \left(2\times\frac{16}{25}-1\right) = \frac{7}{25}.$$

(iii) $\tan 2x = \frac{\sin 2x}{\cos 2x} = \left(\frac{-24}{25}\times\frac{25}{7}\right) = \frac{-24}{7}.$

EXAMPLE 4 (i) *If* $\sin x = \frac{1}{3}$, *find the value of* $\sin 3x$.

(ii) *If* $\cos x = \frac{1}{2}$, *find the value of* $\cos 3x$.

SOLUTION (i) $\sin 3x = (3\sin x - 4\sin^3 x)$

$$= \left\{\left(3\times\frac{1}{3}\right) - 4\times\left(\frac{1}{3}\right)^3\right\} = \left(1-\frac{4}{27}\right) = \frac{23}{27}.$$

(ii) $\cos 3x = (4\cos^3 x - 3\cos x)$

$$= \left\{\left(4\times\frac{1}{8}\right) - \left(3\times\frac{1}{2}\right)\right\} = \left(\frac{1}{2}-\frac{3}{2}\right) = -1.$$

EXAMPLE 5 *If* $\cos x = \frac{4}{5}$ *and* x *is acute, find the value of* $\tan 2x$.

SOLUTION $\cos x = \frac{4}{5} \Rightarrow \sec x = \frac{5}{4}$

$$\Rightarrow \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16}-1} = \sqrt{\frac{9}{16}} = \frac{3}{4}.$$

$$\therefore \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{\left(2\times\frac{3}{4}\right)}{\left(1-\frac{9}{16}\right)} = \left(\frac{3}{2}\times\frac{16}{7}\right) = \frac{24}{7}.$$

EXAMPLE 6 *If* $\tan x = \frac{1}{7}$ *and* $\tan y = \frac{1}{3}$, *show that* $\cos 2x = \sin 4y$.

SOLUTION We have

$$\cos 2x = \frac{(1-\tan^2 x)}{(1+\tan^2 x)} = \frac{\left(1-\frac{1}{49}\right)}{\left(1+\frac{1}{49}\right)} = \left(\frac{48}{49} \times \frac{49}{50}\right) = \frac{24}{25}.$$

$$\sin 4y = 2\sin 2y \cos 2y$$

$$= 2 \times \frac{(2\tan y)}{(1+\tan^2 y)} \times \frac{(1-\tan^2 y)}{(1+\tan^2 y)}$$

$$= \frac{2 \times \left(2 \times \frac{1}{3}\right)}{\left(1+\frac{1}{9}\right)} \times \frac{\left(1-\frac{1}{9}\right)}{\left(1+\frac{1}{9}\right)} = \left(\frac{4}{3} \times \frac{9}{10} \times \frac{8}{9} \times \frac{9}{10}\right) = \frac{24}{25}.$$

$\therefore \quad \cos 2x = \sin 4y$.

EXAMPLE 7 *Prove that*

(i) $\sin \frac{\pi}{6} \cos \frac{\pi}{6} = \frac{\sqrt{3}}{4}$ *(ii)* $\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12} = \frac{\sqrt{3}}{2}$

SOLUTION (i) $\sin \frac{\pi}{6} \cos \frac{\pi}{6} = \frac{1}{2}\left(2\sin \frac{\pi}{6} \cos \frac{\pi}{6}\right)$

$$= \frac{1}{2} \times \sin\left(2 \times \frac{\pi}{6}\right) \qquad [\because\ 2\sin x \cos x = \sin 2x]$$

$$= \frac{1}{2} \sin \frac{\pi}{3} = \left(\frac{1}{2} \times \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}.$$

(ii) $\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12} = \cos\left(2 \times \frac{\pi}{12}\right) \qquad [\because\ \cos^2 x - \sin^2 x = \cos 2x]$

$$= \cos \frac{\pi}{6} = \frac{3}{2}.$$

EXAMPLE 8 *Prove that*

(i) $\frac{\sin 2x}{1-\cos 2x} = \cot x$ *(ii)* $\frac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$

SOLUTION (i) LHS $= \frac{\sin 2x}{1-\cos 2x}$

$$= \frac{2\sin x \cos x}{2\sin^2 x}$$

$$[\because\ \sin 2x = 2\sin x \cos x,\ (1-\cos 2x) = 2\sin^2 x]$$

$= \cot x =$ RHS.

Hence, $\frac{\sin 2x}{1-\cos 2x} = \cot x$.

(ii) LHS $= \dfrac{1 - \cos 2x}{1 + \cos 2x} = \dfrac{2\sin^2 x}{2\cos^2 x}$ $\left[\because \begin{array}{l}(1 - \cos 2x) = 2\sin^2 x, \\ (1 + \cos 2x) = 2\cos^2 x\end{array}\right]$

$= \tan^2 x$ = RHS.

Hence, $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x.$

EXAMPLE 9 *Prove that* $\cos 4x = 1 - 8\sin^2 x \cos^2 x.$

SOLUTION We have

LHS $= \cos 4x$

$= \cos 2(2x) = \cos 2\theta$, where $2x = \theta$

$= \cos^2\theta - \sin^2\theta$ $[\because \cos 2\theta = \cos^2\theta - \sin^2\theta]$

$= (1 - 2\sin^2\theta) = \{1 - 2(\sin 2x)^2\}$

$= 1 - 2(2\sin x \cos x)^2$ $[\because \sin 2x = 2\sin x \cos x]$

$= 1 - 8\sin^2 x \cos^2 x$ = RHS.

Hence, $\cos 4x = 1 - 8\sin^2 x \cos^2 x.$

EXAMPLE 10 *Prove that* $\dfrac{1 + \sin 2x - \cos 2x}{1 + \sin 2x + \cos 2x} = \tan x.$

SOLUTION We have

LHS $= \dfrac{1 + \sin 2x - \cos 2x}{1 + \sin 2x + \cos 2x}$

$= \dfrac{(1 - \cos 2x) + \sin 2x}{(1 + \cos 2x) + \sin 2x}$

$= \dfrac{2\sin^2 x + 2\sin x \cos x}{2\cos^2 x + 2\sin x \cos x}$

$\left[\begin{array}{l}\because (1 - \cos 2x) = 2\sin^2 x, (1 + \cos 2x) = 2\cos^2 x \\ \text{and } \sin 2x = 2\sin x \cos x\end{array}\right]$

$= \dfrac{2\sin x(\sin x + \cos x)}{2\cos x(\sin x + \cos x)} = \tan x$ = RHS.

EXAMPLE 11 *Prove that* $\dfrac{1 - \sin 2x}{1 + \sin 2x} = \tan^2\left(\dfrac{\pi}{4} - x\right).$

SOLUTION We have

LHS $= \dfrac{1 - \sin 2x}{1 + \sin 2x}$

$= \dfrac{1 - \cos\left(\dfrac{\pi}{2} - 2x\right)}{1 + \cos\left(\dfrac{\pi}{2} - 2x\right)}$ $\left[\because \sin 2x = \cos\left(\dfrac{\pi}{2} - 2x\right)\right]$

$$= \frac{2\sin^2\left(\frac{\pi}{4} - x\right)}{2\cos^2\left(\frac{\pi}{4} - x\right)} = \tan^2\left(\frac{\pi}{4} - x\right) = \text{RHS.}$$

$$[\because\ (1 - \cos 2\theta) = 2\sin^2\theta \text{ and } (1 + \cos 2\theta) = 2\cos^2\theta]$$

$$\therefore \quad \frac{1 - \sin 2x}{1 + \sin 2x} = \tan^2\left(\frac{\pi}{4} - x\right).$$

EXAMPLE 12 *Prove that* $\frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x.$

SOLUTION We have

$$\text{LHS} = \frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x}$$

$$= \frac{(\cos x + \sin x)^2 - (\cos x - \sin x)^2}{\cos^2 x - \sin^2 x}$$

$$= \frac{4\sin x \cos x}{\cos^2 x - \sin^2 x} = \frac{2\sin 2x}{\cos 2x}$$

$$= 2\tan 2x = \text{RHS.}$$

EXAMPLE 13 *Prove that* $\tan 4\theta = \frac{4\tan\theta(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}.$

SOLUTION We have

$$\text{LHS} = \tan 4\theta = \tan 2(2\theta)$$

$$= \tan 2x, \text{ where } x = 2\theta$$

$$= \frac{2\tan x}{1 - \tan^2 x} = \frac{2\tan 2\theta}{1 - \tan^2 2\theta}$$

$$= \frac{2 \times \frac{2\tan\theta}{(1 - \tan^2\theta)}}{1 - \left(\frac{2\tan\theta}{1 - \tan^2\theta}\right)^2} = \frac{4\tan\theta(1 - \tan^2\theta)}{(1 - \tan^2\theta)^2 - 4\tan^2\theta}$$

$$= \frac{4\tan\theta(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta} = \text{RHS.}$$

EXAMPLE 14 *Prove that* $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta.$

SOLUTION We have

$$\text{LHS} = \frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta}$$

$$= \frac{\left(\frac{\sin 5\theta}{\cos 5\theta} + \frac{\sin 3\theta}{\cos 3\theta}\right)}{\left(\frac{\sin 5\theta}{\cos 5\theta} - \frac{\sin 3\theta}{\cos 3\theta}\right)} = \frac{(\sin 5\theta \cos 3\theta + \cos 5\theta \sin 3\theta)}{(\sin 5\theta \cos 3\theta - \cos 5\theta \sin 3\theta)}$$

$$= \frac{\sin(5\theta + 3\theta)}{\sin(5\theta - 3\theta)} = \frac{\sin 8\theta}{\sin 2\theta} = \frac{2\sin 4\theta \cos 4\theta}{\sin 2\theta}$$

$$= \frac{4\sin 2\theta \cos 2\theta \cos 4\theta}{\sin 2\theta} = 4\cos 2\theta \cos 4\theta = \text{RHS.}$$

EXAMPLE 15 *Prove that* $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}$.

SOLUTION We have

$$\text{LHS} = \frac{\sec 8\theta - 1}{\sec 4\theta - 1}$$

$$= \frac{\left(\frac{1}{\cos 8\theta} - 1\right)}{\left(\frac{1}{\cos 4\theta} - 1\right)} = \frac{(1 - \cos 8\theta)}{(1 - \cos 4\theta)} \cdot \frac{\cos 4\theta}{\cos 8\theta}$$

$$= \frac{(2\sin^2 4\theta)(\cos 4\theta)}{(2\sin^2 2\theta)(\cos 8\theta)}$$

$$= \frac{(2\sin 4\theta \cos 4\theta)(\sin 4\theta)}{(2\sin^2 2\theta)(\cos 8\theta)}$$

$$= \frac{(\sin 8\theta)(2\sin 2\theta \cos 2\theta)}{(\cos 8\theta)(2\sin^2 2\theta)}$$

$$= \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{RHS.}$$

EXAMPLE 16 *Show that* $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = 2\cos\theta$.

SOLUTION $\text{LHS} = \sqrt{2 + \sqrt{2 + 2\cos 4\theta}}$

$$= \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} = \sqrt{2 + \sqrt{4\cos^2 2\theta}}$$

$$= \sqrt{2 + 2\cos 2\theta} = \sqrt{2(1 + \cos 2\theta)}$$

$$= \sqrt{4\cos^2\theta} = 2\cos\theta = \text{RHS.}$$

EXAMPLE 17 *Prove that* $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$.

SOLUTION We have

$$\cos 5x = \cos(3x + 2x)$$

$$= \cos 3x \cos 2x - \sin 3x \sin 2x$$

$$= (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (3\sin x - 4\sin^3 x)(2\sin x \cos x)$$

$$= (8\cos^5 x - 10\cos^3 x + 3\cos x) - 6\sin^2 x \cos x + 8\sin^4 x \cos x$$

$$= (8\cos^5 x - 10\cos^3 x + 3\cos x) - 6(1-\cos^2 x)\cos x + 8(1-\cos^2 x)^2 \cdot \cos x$$

$$= 16\cos^5 x - 20\cos^3 x + 5\cos x.$$

Hence, $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$.

EXAMPLE 18 *Prove that* $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

SOLUTION We have

$$\cos 6x = \cos 2(3x)$$

$$= \cos 2\theta, \text{ where } 3x = \theta$$

$$= 2\cos^2\theta - 1 = 2\cos^2 3x - 1$$

$$= 2(4\cos^3 x - 3\cos x)^2 - 1 \quad [\because \cos 3x = (4\cos^3 x - 3\cos x)]$$

$$= 2(9\cos^2 x + 16\cos^6 x - 24\cos^4 x) - 1$$

$$= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1.$$

Hence, $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

EXAMPLE 19 *Prove that* $\cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1$.

SOLUTION We have

$$\cot 3x = \cot(2x + x)$$

$$\Leftrightarrow \quad \cot 3x = \frac{\cot 2x \cot x - 1}{\cot 2x + \cot x}$$

$$\Leftrightarrow \quad \cot 3x \cot 2x + \cot 3x \cot x = \cot 2x \cot x - 1$$

$$\Leftrightarrow \quad \cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1.$$

Hence, $\cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1$.

EXAMPLE 20 *Find the value of*

(i) $\sin 18°$ *(ii)* $\cos 18°$ *(iii)* $\cos 36°$ *(iv)* $\sin 36°$

(v) $\sin 72°$ *(vi)* $\cos 72°$ *(vii)* $\sin 54°$ *(viii)* $\cos 54°$

SOLUTION (i) Let $\theta = 18°$. Then,

$$\theta = 18° \Rightarrow 5\theta = 90°$$

$$\Rightarrow 2\theta = (90° - 3\theta)$$

$$\Rightarrow \sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta$$

$$\Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$$

$$\Rightarrow 2\sin\theta\cos\theta - 4\cos^3\theta + 3\cos\theta = 0$$

$$\Rightarrow \cos\theta(2\sin\theta - 4\cos^2\theta + 3) = 0$$

$$\Rightarrow 2\sin\theta - 4\cos^2\theta + 3 = 0 \quad [\because \cos\theta = \cos 18° \neq 0]$$

$$\Rightarrow 2\sin\theta - 4(1 - \sin^2\theta) + 3 = 0$$

$$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\Rightarrow \sin\theta = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{(-1 \pm \sqrt{5})}{4}$$

$$\Rightarrow \sin\theta = \frac{(\sqrt{5} - 1)}{4} \quad [\because \sin\theta = \sin 18° > 0].$$

$$\therefore \quad \sin 18° = \frac{(\sqrt{5} - 1)}{4}.$$

(ii) $\cos^2 18° = (1 - \sin^2 18°)$

$$= \left\{1 - \frac{(\sqrt{5}-1)^2}{16}\right\} = \left\{1 - \frac{(6-2\sqrt{5})}{16}\right\} = \frac{10+2\sqrt{5}}{16}$$

$$\Leftrightarrow \quad \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} \qquad [\because \ \cos 18° > 0].$$

(iii) $\cos 36° = (1 - 2\sin^2 18°)$

$$= \left\{1 - 2\cdot\frac{(\sqrt{5}-1)^2}{16}\right\} = \left\{1 - \frac{(6-2\sqrt{5})}{8}\right\}$$

$$= \frac{\sqrt{5}+1}{4}.$$

(iv) $\sin 36° = \sqrt{1-\cos^2 36°} = \left\{1 - \frac{(\sqrt{5}+1)^2}{16}\right\}^{½}$

$$= \left\{\frac{10-2\sqrt{5})}{16}\right\}^{½} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

(v) $\sin 72° = \sin\ (90° - 18°) = \cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}.$

(vi) $\cos 72° = \cos\ (90° - 18°) = \sin 18° = \dfrac{(\sqrt{5}-1)}{4}.$

(vii) $\sin 54° = \sin\ (90° - 36°) = \cos 36° = \dfrac{(\sqrt{5}+1)}{4}.$

(viii) $\cos 54° = \cos\ (90° - 36°) = \sin\ 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}.$

SUMMARY

(i) $\sin 18° = \dfrac{(\sqrt{5}-1)}{4} = \cos 72°$

(ii) $\cos 18° = \dfrac{\sqrt{(10+2\sqrt{5})}}{4} = \sin 72°$

(iii) $\cos 36° = \dfrac{(\sqrt{5}+1)}{4} = \sin 54°$

(iv) $\sin 36° = \dfrac{\sqrt{(10-2\sqrt{5})}}{4} = \cos 54°$

EXAMPLE 21 *Prove that* $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10} = \dfrac{-1}{2}.$

SOLUTION We have

$$\text{LHS} = \sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$$

$$= \sin\frac{\pi}{10} + \sin\left(\pi + \frac{3\pi}{10}\right) = \sin\frac{\pi}{10} - \sin\frac{3\pi}{10} \quad [\because \sin(\pi + \theta) = -\sin\theta]$$

$$= \sin 18^\circ - \sin 54^\circ = \sin 18^\circ - \sin(90^\circ - 36^\circ)$$

$$= (\sin 18^\circ - \cos 36^\circ) = \left\{\frac{(\sqrt{5}-1)}{4} - \frac{(\sqrt{5}+1)}{4}\right\} = \frac{-1}{2}$$

$$\left[\because \sin 18^\circ = \frac{(\sqrt{5}-1)}{4} \text{ and } \cos 36^\circ = \frac{(\sqrt{5}+1)}{4}\right]$$

EXAMPLE 22 *Prove that* $(\cos^2 48^\circ - \sin^2 12^\circ) = \frac{(\sqrt{5}+1)}{8}$.

SOLUTION We have

$$\text{LHS} = (\cos^2 48^\circ - \sin^2 12^\circ)$$

$$= \frac{1}{2}(2\cos^2 48^\circ - 2\sin^2 12^\circ)$$

$$= \frac{1}{2}\{(1 + \cos 96^\circ) - (1 - \cos 24^\circ)\}$$

$$[\because 2\cos^2\theta = (1 + \cos 2\theta) \text{ and } 2\sin^2\theta = (1 - \cos 2\theta)]$$

$$= \frac{1}{2}(\cos 96^\circ + \cos 24^\circ)$$

$$= \frac{1}{2}\left[2\cos\left(\frac{96^\circ + 24^\circ}{2}\right)\cos\left(\frac{96^\circ - 24^\circ}{2}\right)\right]$$

$$= \frac{1}{2} \times 2\cos 60^\circ \cos 36^\circ = \frac{1}{2} \times \frac{(\sqrt{5}+1)}{4} \quad \left[\because \cos 36^\circ = \frac{(\sqrt{5}+1)}{4}\right]$$

$$= \frac{(\sqrt{5}+1)}{8} = \text{RHS.}$$

EXAMPLE 23 *Prove that* $(\sin^2 72^\circ - \sin^2 60^\circ) = \frac{(\sqrt{5}-1)}{8}$.

SOLUTION We have

$$\text{LHS} = (\sin^2 72^\circ - \sin^2 60^\circ)$$

$$= \frac{1}{2}(2\sin^2 72^\circ - 2\sin^2 60^\circ)$$

$$= \frac{1}{2}\{(1 - \cos 144^\circ) - (1 - \cos 120^\circ)\} \quad [\because 2\sin^2\theta = (1 - \cos 2\theta)]$$

$$= \frac{1}{2}(\cos 120^\circ - \cos 144^\circ)$$

$$= -\frac{1}{4} - \frac{1}{2}\cos 144^\circ = -\frac{1}{4} - \frac{1}{2}\cos(180^\circ - 36^\circ) \quad \left[\because \cos 120^\circ = -\frac{1}{2}\right]$$

$$= -\frac{1}{4} + \frac{1}{2}\cos 36^\circ = -\frac{1}{4} + \frac{(\sqrt{5}+1)}{8} \quad \left[\because \cos 36^\circ = \frac{(\sqrt{5}+1)}{4}\right]$$

$$= \frac{(\sqrt{5}-1)}{8} = \text{RHS.}$$

EXAMPLE 24 *Prove that* $\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ \\
&= \frac{1}{4}(2\cos 66^\circ \cos 6^\circ)(2\cos 78^\circ \cos 42^\circ) \\
&= \frac{1}{4}[\cos(66^\circ + 6^\circ) + \cos(66^\circ - 6^\circ)] \times [\cos(78^\circ + 42^\circ) + \cos(78^\circ - 42^\circ)] \\
&= \frac{1}{4}(\cos 72^\circ + \cos 60^\circ)(\cos 120^\circ + \cos 36^\circ) \\
&= \frac{1}{4}\left(\sin 18^\circ + \frac{1}{2}\right)\left(-\frac{1}{2} + \cos 36^\circ\right) \quad [\because \cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ] \\
&= \frac{1}{4}\left[\frac{(\sqrt{5}-1)}{4} + \frac{1}{2}\right]\left[-\frac{1}{2} + \frac{(\sqrt{5}+1)}{4}\right] \quad \left[\because \sin 18^\circ = \frac{(\sqrt{5}-1)}{4} \text{ and } \cos 36^\circ = \frac{(\sqrt{5}+1)}{4}\right] \\
&= \frac{1}{4} \cdot \frac{(\sqrt{5}+1)}{4} \cdot \frac{(\sqrt{5}-1)}{4} = \frac{(5-1)}{64} \\
&= \frac{4}{64} = \frac{1}{16} = \text{RHS.}
\end{aligned}$$

EXAMPLE 25 *Prove that* $\sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5} = \frac{5}{16}$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5} \\
&= \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin\left(\pi - \frac{2\pi}{5}\right) \sin\left(\pi - \frac{\pi}{5}\right) \\
&= \sin^2 \frac{\pi}{5} \sin^2 \frac{2\pi}{5} \quad [\because \sin(\pi - \theta) = \sin\theta] \\
&= (\sin 36^\circ)^2 \times (\sin 72^\circ)^2 \\
&= (\sin 36^\circ)^2 \times (\cos 18^\circ)^2 \quad [\because \sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ] \\
&= \frac{(10 - 2\sqrt{5})}{16} \times \frac{(10 + 2\sqrt{5})}{16} = \frac{(100 - 20)}{(16 \times 16)} \quad \left[\because \sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4} \text{ and } \cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}\right] \\
&= \frac{80}{(16 \times 16)} = \frac{5}{16} = \text{RHS.}
\end{aligned}$$

EXAMPLE 26 *Find the value of*

(i) sin 22° 30′ *(ii)* cos 22° 30′ *(iii)* tan 22° 30′

SOLUTION (i) $\sin^2\theta = \dfrac{(1-\cos 2\theta)}{2}$

$$\Rightarrow \sin^2(22^\circ\ 30') = \frac{(1-\cos 45^\circ)}{2} = \frac{\left(1-\frac{1}{\sqrt{2}}\right)}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$\Rightarrow \sin(22^\circ\ 30') = \sqrt{\frac{(\sqrt{2}-1)}{2\sqrt{2}}}.$$

(ii) $\cos^2\theta = \dfrac{(1+\cos 2\theta)}{2}$

$$\Rightarrow \cos^2(22^\circ\ 30') = \frac{(1+\cos 45^\circ)}{2} = \frac{\left(1+\frac{1}{\sqrt{2}}\right)}{2} = \frac{(\sqrt{2}+1)}{2\sqrt{2}}$$

$$\Rightarrow \cos(22^\circ\ 30') = \sqrt{\frac{(\sqrt{2}+1)}{2\sqrt{2}}}.$$

(iii) $$\tan^2(22^\circ\ 30') = \frac{\sin^2(22^\circ\ 30')}{\cos^2(22^\circ\ 30')} = \frac{(\sqrt{2}-1)}{(2\sqrt{2})} \times \frac{(2\sqrt{2})}{(\sqrt{2}+1)}$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)}{(\sqrt{2}+1)} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$$

$$= (\sqrt{2}-1)^2$$

$$\Rightarrow \tan(22^\circ\ 30') = (\sqrt{2}-1).$$

EXERCISE 15D

1. If $\sin x = \dfrac{\sqrt{5}}{3}$ and $0 < x < \dfrac{\pi}{2}$, find the values of

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

2. If $\cos x = \dfrac{-3}{5}$ and $\pi < x < \dfrac{3\pi}{2}$, find the values of

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

3. If $\tan x = \dfrac{-5}{12}$ and $\dfrac{\pi}{2} < x < \pi$, find the values of

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

4. (i) If $\sin x = \dfrac{1}{6}$, find the value of $\sin 3x$.

(ii) If $\cos x = \dfrac{-1}{2}$, find the value of $\cos 3x$.

Prove that

5. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

6. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

7. $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

8. $\dfrac{\tan 2x}{1 + \sec 2x} = \tan x$

9. $\sin 2x(\tan x + \cot x) = 2$

10. $\operatorname{cosec} 2x + \cot 2x = \cot x$

11. $\cos 2x + 2\sin^2 x = 1$

12. $(\sin x - \cos x)^2 = 1 - \sin 2x$

13. $\cot x - 2\cot 2x = \tan x$

14. $(\cos^4 x + \sin^4 x) = \dfrac{1}{2}(2 - \sin^2 2x)$

15. $\dfrac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = \dfrac{1}{2}(2 + \sin 2x)$

16. $\dfrac{1 - \cos 2x + \sin x}{\sin 2x + \cos x} = \tan x$

17. $\cos x \cos 2x \cos 4x \cos 8x = \dfrac{\sin 16x}{16\sin x}$

18. Prove that

(i) $2\sin 22\dfrac{1^\circ}{2} \cos 22\dfrac{1^\circ}{2} = \dfrac{1}{\sqrt{2}}$ (ii) $2\cos^2 15^\circ - 1 = \dfrac{\sqrt{3}}{2}$

(iii) $8\cos^3 20^\circ - 6\cos 20^\circ = 1$ (iv) $3\sin 40^\circ - 4\sin^3 40^\circ = \dfrac{\sqrt{3}}{2}$

19. Prove that

(i) $\sin^2 24^\circ - \sin^2 6^\circ = \dfrac{(\sqrt{5} - 1)}{8}$ (ii) $\sin^2 72^\circ - \cos^2 30^\circ = \dfrac{(\sqrt{5} - 1)}{8}$

20. Prove that $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$

21. If $\tan\theta = \dfrac{a}{b}$, prove that $a\sin 2\theta + b\cos 2\theta = b.$

ANSWERS (EXERCISE 15D)

1. (i) $\dfrac{4\sqrt{5}}{9}$ (ii) $\dfrac{-1}{9}$ (iii) $-4\sqrt{5}$

2. (i) $\dfrac{24}{25}$ (ii) $\dfrac{-7}{25}$ (iii) $\dfrac{-24}{7}$

3. (i) $\dfrac{-120}{169}$ (ii) $\dfrac{119}{169}$ (iii) $\dfrac{-120}{119}$

4. (i) $\dfrac{13}{27}$ (ii) 1

HINTS TO SOME SELECTED QUESTIONS

3. Use the relations $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$ and $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

5. LHS $= \dfrac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \cos x + \sin x =$ RHS.

6. LHS $= \dfrac{2\sin x \cos x}{2\cos^2 x} = \tan x =$ RHS.

7. LHS $= \dfrac{2\sin x \cos x}{2\sin^2 x} = \cot x =$ RHS.

8. LHS $= \dfrac{\sin 2x}{(1+\cos 2x)} = \dfrac{2\sin x \cos x}{2\cos^2 x} = \tan x =$ RHS.

10. LHS $= \dfrac{1}{\sin 2x} + \dfrac{\cos 2x}{\sin 2x} = \dfrac{1+\cos 2x}{\sin 2x} = \dfrac{2\cos^2 x}{2\sin x\cos x} = \cot x =$ RHS.

13. LHS $= \left(\dfrac{1}{\tan x} - \dfrac{2}{\tan 2x}\right) = \left\{\dfrac{1}{\tan x} - \dfrac{2(1-\tan^2 x)}{2\tan x}\right\} = \dfrac{\tan^2 x}{\tan x} = \tan x =$ RHS.

14. LHS $= (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x$

$= 1 - 2\cos^2 x\sin^2 x = 1 - \dfrac{1}{2}(\sin^2 2x) = \dfrac{1}{2}(2 - \sin^2 2x) =$ RHS.

16. LHS $= \dfrac{2\sin^2 x + \sin x}{2\sin x\cos x + \cos x} = \dfrac{\sin x(2\sin x + 1)}{\cos x(2\sin x + 1)} = \dfrac{\sin x}{\cos x} = \tan x =$ RHS.

17. LHS $= \dfrac{1}{2\sin x}(2\sin x\cos x)\cos 2x\cos 4x\cos 8x$

$= \dfrac{1}{2\sin x}(\sin 2x\cos 2x)\cos 4x\cos 8x$

$= \dfrac{1}{4\sin x}(2\sin 2x\cos 2x)\cos 4x\cos 8x$

$= \dfrac{1}{4\sin x}(\sin 4x\cos 4x)\cos 8x$

$= \dfrac{1}{8\sin x}(2\sin 4x\cos 4x)\cos 8x$

$= \dfrac{1}{8\sin x}(\sin 8x\cos 8x)$

$= \dfrac{1}{16\sin x}(2\sin 8x\cos 8x)$

$= \dfrac{1}{16\sin x}(\sin 16x) = \dfrac{\sin 16x}{16\sin x} =$ RHS.

18. (i) LHS $= \sin\left(2\times 22\dfrac{1^\circ}{2}\right) = \sin 45^\circ = \dfrac{1}{\sqrt{2}}$.

(ii) LHS $= (2\cos^2\theta - 1) = \cos 2\theta$, where $\theta = 15^\circ$

$= \cos 30^\circ = \dfrac{\sqrt{3}}{2}$.

(iii) LHS $= 2(4\cos^3\theta - 3\cos\theta)$, where $\theta = 20^\circ$

$= 2\times\cos 3\theta = 2\times\cos 60^\circ = \left(2\times\dfrac{1}{2}\right) = 1.$

(iv) LHS $= (3\sin\theta - 4\sin^3\theta)$, where $\theta = 40^\circ$

$= \sin 3\theta = \sin 120^\circ = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$.

20. LHS $= \dfrac{\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ}{\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ}$.

21. Use the relations

$\sin 2\theta = \dfrac{2\tan\theta}{(1+\tan^2\theta)}$ and $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.

TRIGONOMETRICAL FUNCTIONS OF HALF ANGLES

THEOREM 1 *For all* $x \in R$, *we have*

(i) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ *(ii)* $1 - \cos x = 2\sin^2\frac{x}{2}$

(iii) $1 + \cos x = 2\cos^2\frac{x}{2}$

PROOF (i) $\sin x = \sin\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \sin\frac{x}{2}\cos\frac{x}{2} + \cos\frac{x}{2}\sin\frac{x}{2}$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\therefore \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}.$$

(ii) $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \cos\frac{x}{2}\cos\frac{x}{2} - \sin\frac{x}{2}\sin\frac{x}{2}$$

$$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$= \left(1 - \sin^2\frac{x}{2}\right) - \sin^2\frac{x}{2}$$

$$= \left(1 - 2\sin^2\frac{x}{2}\right).$$

$$\therefore (1 - \cos x) = 2\sin^2\frac{x}{2}.$$

(iii) $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \cos\frac{x}{2}\cos\frac{x}{2} - \sin\frac{x}{2}\sin\frac{x}{2}$$

$$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$= \cos^2\frac{x}{2} - \left(1 - \cos^2\frac{x}{2}\right)$$

$$= \left(2\cos^2\frac{x}{2} - 1\right).$$

$$\therefore (1 + \cos x) = 2\cos^2\frac{x}{2}.$$

THEOREM 2 *For all* $x \in R$, *we have*

(i) $\sin x = \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$ *(ii)* $\cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

PROOF (i) $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)}$$

$$= \frac{\left\{\frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right\}}{1 + \left\{\frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}\right\}} \quad \left[\text{dividing num. and denom. by } \cos^2\left(\frac{x}{2}\right)\right]$$

$$= \frac{2\tan\left(\frac{x}{2}\right)}{\left\{1 + \tan^2\left(\frac{x}{2}\right)\right\}}.$$

$$\therefore \quad \sin x = \frac{2\tan\left(\frac{x}{2}\right)}{\left\{1 + \tan^2\left(\frac{x}{2}\right)\right\}}.$$

(ii) $\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$

$$= \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)} \quad \left[\because\ 1 = \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)\right]$$

$$= \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} \quad \left[\text{dividing num. and denom. by } \cos^2\left(\frac{x}{2}\right)\right].$$

$$\therefore \quad \cos x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}.$$

SUMMARY

(i) $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

(ii) $(1 - \cos x) = 2\sin^2\left(\frac{x}{2}\right)$

(iii) $(1 + \cos x) = 2\cos^2\left(\frac{x}{2}\right)$

(iv) $\sin x = \dfrac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

(v) $\cos x = \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\tan x = \frac{-4}{3}$ *and* $\frac{\pi}{2} < x < \pi$, *find the values of*

(i) $\sin\frac{x}{2}$, *(ii)* $\cos\frac{x}{2}$, *(iii)* $\tan\frac{x}{2}$.

SOLUTION Since x lies in Quadrant II, we have $\cos x < 0$.

$$\because \quad \tan x = \frac{-4}{3} \Rightarrow \sec^2 x = (1 + \tan^2 x) = \left(1 + \frac{16}{9}\right) = \frac{25}{9}$$

$$\Rightarrow \cos^2 x = \frac{1}{\sec^2 x} = \frac{9}{25}$$

$$\Rightarrow \cos x = -\sqrt{\frac{9}{25}} = -\frac{3}{5}.$$

Also, $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\Rightarrow \frac{x}{2}$ lies in Quadrant I

$\Rightarrow \sin\frac{x}{2} > 0$ and $\cos\frac{x}{2} > 0$.

(i) $2\sin^2\frac{x}{2} = (1 - \cos x) = \left(1 + \frac{3}{5}\right) = \frac{8}{5}$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{8}{10} = \frac{4}{5}$$

$$\Rightarrow \sin\frac{x}{2} = +\sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} \quad \left[\because \sin\frac{x}{2} > 0\right].$$

(ii) $2\cos^2\frac{x}{2} = (1 + \cos x) = \left(1 - \frac{3}{5}\right) = \frac{2}{5}$

$\Rightarrow \cos^2\frac{x}{2} = \left(\frac{2}{5 \times 2}\right) = \frac{1}{5}$

$\Rightarrow \cos\frac{x}{2} = +\sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} \quad \left[\because \cos\frac{x}{2} > 0\right].$

(iii) $\tan\frac{x}{2} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \left(\frac{2}{\sqrt{5}} \times \sqrt{5}\right) = 2.$

EXAMPLE 2 *If* $\tan x = \frac{3}{4}$ *and* $\pi < x < \frac{3\pi}{2}$, *find the values of*

(i) $\sin\frac{x}{2}$, *(ii)* $\cos\frac{x}{2}$, *(iii)* $\tan\frac{x}{2}$.

SOLUTION Since x lies in Quadrant III, we have $\cos x < 0$.

Now, $\tan x = \frac{3}{4} \Rightarrow \sec^2 x = (1 + \tan^2 x) = \left(1 + \frac{9}{16}\right) = \frac{25}{16}$

$\Rightarrow \cos^2 x = \frac{1}{\sec^2 x} = \frac{16}{25}$

$\Rightarrow \cos x = -\sqrt{\frac{16}{25}} = \frac{-4}{5}.$

Also, $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

$\Rightarrow \frac{x}{2}$ lies in Quadrant II

$\Rightarrow \sin\frac{x}{2} > 0$ and $\cos\frac{x}{2} < 0.$

(i) $2\sin^2\frac{x}{2} = (1 - \cos x) = \left(1 + \frac{4}{5}\right) = \frac{9}{5}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{9}{10}$

$\Rightarrow \sin\frac{x}{2} = +\sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} \quad \left[\because \sin\frac{x}{2} > 0\right].$

(ii) $2\cos^2\frac{x}{2} = (1 + \cos x) = \left(1 - \frac{4}{5}\right) = \frac{1}{5}$

$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{10}$

$\Rightarrow \cos\frac{x}{2} = -\sqrt{\frac{1}{10}} = \frac{-1}{\sqrt{10}} \quad \left[\because \cos\frac{x}{2} < 0\right].$

(iii) $\tan\frac{x}{2} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \left(\frac{3}{\sqrt{10}} \times \frac{\sqrt{10}}{-1}\right) = -3.$

EXAMPLE 3 *If* $\cos x = -\frac{1}{3}$ *and* x *lies in Quadrant III, find the values of*

(i) $\sin\frac{x}{2}$, *(ii)* $\cos\frac{x}{2}$, *(iii)* $\tan\frac{x}{2}$.

SOLUTION Since x lies in Quadrant III, we have

$$\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

$$\Rightarrow \frac{x}{2} \text{ lies in Quadrant II}$$

$$\Rightarrow \sin\frac{x}{2} > 0 \text{ and } \cos\frac{x}{2} < 0.$$

(i) $2\sin^2\frac{x}{2} = (1 - \cos x) = \left(1 + \frac{1}{3}\right) = \frac{4}{3}$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{2}{3}$$

$$\Rightarrow \sin\frac{x}{2} = +\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \quad \left[\because \sin\frac{x}{2} > 0\right].$$

(ii) $2\cos^2\frac{x}{2} = (1 + \cos x) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{3}$$

$$\Rightarrow \cos\frac{x}{2} = -\frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

(iii) $\tan\frac{x}{2} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \left(\frac{\sqrt{6}}{3} \times \frac{3}{-\sqrt{3}}\right) = -\sqrt{2}.$

EXAMPLE 4 *Prove that* $\frac{1 + \cos x}{1 - \cos x} = (\text{cosec } x + \cot x)^2$.

SOLUTION We have

$$\text{LHS} = \frac{1 + \cos x}{1 - \cos x}$$

$$= \frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)} = \cot^2\frac{x}{2} \quad \left[\because 1 + \cos x = 2\cos^2\left(\frac{x}{2}\right),\; 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)\right]$$

$$\text{RHS} = (\text{cosec}\, x + \cot x)^2$$

$$= \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)^2 = \left(\frac{1 + \cos x}{\sin x}\right)^2$$

$$= \left\{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right\}^2 = \left\{\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right\}^2 = \cot^2\frac{x}{2}.$$

$\therefore$ LHS = RHS.

EXAMPLE 5 *Prove that* $\dfrac{\cos x}{(1 - \sin x)} = \tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right).$

SOLUTION We have

$$\text{LHS} = \frac{\cos x}{(1 - \sin x)}$$

$$= \frac{\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}{\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)}$$

$$\left[\because \cos x = \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right), \cos^2\frac{x}{2} + \sin^2\frac{x}{2} = 1 \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}$$

$$= \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)} = \frac{\left(1 + \tan\frac{x}{2}\right)}{\left(1 - \tan\frac{x}{2}\right)} = \frac{\left(\tan\frac{\pi}{4} + \tan\frac{x}{2}\right)}{\left(1 - \tan\frac{\pi}{4}\cdot\tan\frac{x}{2}\right)}$$

$$\left[\text{dividing num. and denom. by } \cos\frac{x}{2}\right]$$

$$= \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \text{RHS}.$$

EXAMPLE 6 *If* α *and* β *be two distinct real numbers such that* $(\alpha - \beta) \neq 2n\pi$ *for any intetger n, satisfying the equation* $a\cos\theta + b\sin\theta = c$ *then prove that*

(i) $\cos(\alpha + \beta) = \dfrac{a^2 - b^2}{a^2 + b^2}$ *(ii)* $\sin(\alpha + \beta) = \dfrac{2ab}{a^2 + b^2}.$

SOLUTION Since α and β satisfy the equation $a\cos\theta + b\sin\theta = c$, we have

$a\cos\alpha + b\sin\alpha = c,$ ... (i)

$a\cos\beta + b\sin\beta = c.$ ... (ii)

Subtracting (ii) from (i), we get

$$a(\cos\alpha - \cos\beta) + b(\sin\alpha - \sin\beta) = 0$$

$$\Rightarrow \quad -2a\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + 2b\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = 0$$

$$\Rightarrow \quad -2\sin\left(\frac{\alpha-\beta}{2}\right)\left[a\sin\left(\frac{\alpha+\beta}{2}\right) - b\cos\left(\frac{\alpha+\beta}{2}\right)\right] = 0$$

$$\Rightarrow \quad a\sin\left(\frac{\alpha+\beta}{2}\right) - b\cos\left(\frac{\alpha+\beta}{2}\right) = 0$$

$$\left[\because (\alpha-\beta) \neq 2n\pi \Rightarrow \left(\frac{\alpha-\beta}{2}\right) \neq n\pi \quad \therefore \sin\left(\frac{\alpha-\beta}{2}\right) \neq \sin n\pi \neq 0\right]$$

$$\Rightarrow \quad \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a}. \qquad \ldots \text{(iii)}$$

(i) $$\cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} \qquad \left[\because \cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{\left(1-\frac{b^2}{a^2}\right)}{\left(1+\frac{b^2}{a^2}\right)} = \frac{a^2-b^2}{a^2+b^2} \quad \text{[using (iii)].}$$

(ii) $$\sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} \qquad \left[\because \sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{2\left(\frac{b}{a}\right)}{\left(1+\frac{b^2}{a^2}\right)} = \frac{2ab}{a^2+b^2} \quad \text{[using (iii)].}$$

EXERCISE 15E

1. If $\sin x = \frac{\sqrt{5}}{3}$ and $\frac{\pi}{2} < x < \pi$, find the values of

(i) $\sin\frac{x}{2}$, (ii) $\cos\frac{x}{2}$, (iii) $\tan\frac{x}{2}$.

2. If $\cos x = \frac{-3}{5}$ and $\frac{\pi}{2} < x < \pi$, find the values of

(i) $\sin\frac{x}{2}$, (ii) $\cos\frac{x}{2}$, (iii) $\tan\frac{x}{2}$.

3. If $\sin x = \frac{-1}{2}$ and x lies in Quadrant IV, find the values of

(i) $\sin \frac{x}{2}$, (ii) $\cos \frac{x}{2}$, (iii) $\tan \frac{x}{2}$.

4. If $\cos \frac{x}{2} = \frac{12}{13}$ and x lies in Quadrant I, find the values of

(i) $\sin x$, (ii) $\cos x$, (iii) $\cot x$.

5. If $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find the value of $\tan \frac{x}{2}$.

Prove that

6. $\cot \frac{x}{2} - \tan \frac{x}{2} = 2\cot x$

7. $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tan x + \sec x$

8. $\sqrt{\frac{1+\sin x}{1-\sin x}} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

9. $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = 2\sec x$

10. $\frac{\sin x}{1+\cos x} = \tan \frac{x}{2}$

ANSWERS (EXERCISE 15E)

1. (i) $\frac{\sqrt{30}}{6}$ (ii) $\frac{\sqrt{6}}{6}$ (iii) $\sqrt{5}$ 2. (i) $\frac{2\sqrt{5}}{5}$ (ii) $\frac{1}{\sqrt{5}}$ (iii) 2

3. (i) $\sin \frac{x}{2} = \frac{\sqrt{(2-\sqrt{3})}}{2}$ (ii) $\cos \frac{x}{2} = -\frac{\sqrt{(2+\sqrt{3})}}{2}$ (iii) $\tan \frac{x}{2} = -(2-\sqrt{3})$

4. (i) $\frac{120}{169}$ (ii) $\frac{119}{169}$ (iii) $\frac{119}{120}$ 5. $\frac{1}{3}$

HINTS TO SOME SELECTED QUESTIONS

5. $\tan \frac{x}{2} = \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$, where $\cos x = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.

6. $$\text{LHS} = \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} - \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \frac{\left\{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right\}}{\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \frac{2\cos x}{\sin x} = 2\cot x = \text{RHS}.$$

7. RHS $= \left(\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}\right) = \dfrac{(1+\sin x)}{\cos x} = \dfrac{1-\cos\left(\dfrac{\pi}{2}+x\right)}{\sin\left(\dfrac{\pi}{2}+x\right)}$

$$= \frac{2\sin^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{2\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \text{LHS.}$$

8. LHS $= \dfrac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}} \times \dfrac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}} = \dfrac{(1+\sin x)}{\cos x} = \dfrac{1-\cos\left(\dfrac{\pi}{2}+x\right)}{\sin\left(\dfrac{\pi}{2}+x\right)}$

$$= \frac{2\sin^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{2\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \text{RHS.}$$

10. LHS $= \dfrac{\sin x}{(1+\cos x)} = \dfrac{2\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)}{2\cos^2\left(\dfrac{x}{2}\right)} = \tan\dfrac{x}{2} = \text{RHS.}$